Brain Ticklers

RESULTS FROM SUMMER

Perfect

Antmann, Eric D.	FL	B	'14
Buckley, Robert C.	TN	Z	'12
*Gibbs, Kenneth P.	MO	Γ	'76
*Harter, Eamonn T.	ID	Γ	'06
*Harvey, Arthur J.	OH	A	'83
McGraw, Sean	Husband of member		
Norris, Thomas G.	OK	A	'56
Quan, Richard	CA	A	'01
Richards, John R.	NJ	B	'76
Schmidt, V. Hugo	WA	B	'51
Spong, Robert N.	UT	A	'58
*Stribling, Jeffrey R.	CA	A	'92
*Strong, Michael D.	PA	A	'84
*Van Wyk, Rogell	IN	A	'59
*Wendling, D. Greg	IL	A	'79

Other

Allen, Jonathan M.	MI	Γ	'92
Aron, Gert	IA	B	'58
*Bohdan, Timothy E.	IN	Γ	'85
Cirilli, John S.	MN	B	'05
Eckley, Paul L.	NV	A	'75
Edge, Billy L.	GA	A	'71
Handley, Vernon K.	GA	A	'86
Hasek, William R.	PA	Γ	'49
Jones, Donlan F.	CA	Z	'52
Joy, Robert L.	CA	N	'66
*Kimmel, Peter G.	Husband of member		
McCormick, Raynard	Non-member		
Rentz, Peter E.	IN	A	'55
Ricketts-Greene, Janique	DC	A	'01
Rubin, James D.	MI	Γ	'82
Shah, Parth	Son of member		
Shamblin, G. Richard	FL	A	'72
Stadlin, Walter O.	NJ	Γ	'52
Summerfield, Steven L.	MO	Γ	'85
Surrey, Robert I.	NY	Δ	'72
Sutor, David C.	Son of member		
*Thaller, David B.	MA	B	'93
Vinoski, Stephen B.	TN	Δ	'85
Voellinger, Edward J.	Non-member		

*Denotes correct bonus solution

SUMMER REVIEW

Problems 3 (game scores) and 5 (cryptic addition) were the ones keeping people from getting perfect scores. Several people only got half credit to the Bonus (cyclic towers), since they did not answer both parts correctly.

FALL SOLUTIONS

Readers' entries for the Fall problems will be acknowledged in the Spring '14 *Bent*. Meanwhile, here are the answers:

1 The Red Lion is **13 miles** from the Purple Cow. In the first interval, with Bob walking and Carl cycling, we have $4/W_B = (D-4)/C_C$, or $W_B/C_C = 4/(D-4)$, where D is the distance between the towns, and C and W are cycling and walking speeds. The third interval is a similar situation to the first, with Bob walking and Carl cycling, so $(D-7+2)/W_B = (D+7-2)/C_C$, or $W_B/C_C = (D-5)/(D+5)$. Setting these expressions equal gives $4/(D-4) = (D-5)/(D+5)$, which can be solved for D to give $D = 13$. The other information provided is just a red herring.

2 Ann has **13 Canadian stamps** and **4 French stamps.** France and Canada are most often mentioned, so express all the others in terms of F or C. This gives $J = 2C$; $D = 10F$; $E = 5C - 10$; $H = 5F$; $L = C - 2$; $I = 3C$; $A = (3C - 1)/2$; $K = 3C - 1$; $B = 4F$; and $G = 2C - 4$; which, along with F and C, sums to $20F + 18.5C - 17.5 = 303$ or $40F + 37C = 641$. So C must be odd and must be 1, 3, 5, 7, 9, 11, 13, or 15. Only $C = 13$ gives an integral value for F, namely $F = 4$.

3 The four smallest N values are **1910, 9110, 89,998,** and **98,998,** which are unique solutions for A + B = AC; A + B = BC; BA + A = AB; and A + BA = AB. You can find these with the help of a computer or patiently try all possible four and five digit cryptic additions manually.

4 The probability that a randomly chosen positive integer N has no repeated prime factors is $P = 6/\pi^2 =$ **0.60793.** The probability that N has the prime factor p is $1/p$ (every pth integer is divisible by p), and the probability that it has the factor p^2 is $1/p^2$. Thus, the probability that N is not divisible by p^2 is $1 - 1/p^2$, and the probability P that it has no repeated prime factors is $P = (1 - 1/p_1^2)(1 - 1/p_2^2)(1 - 1/p_3^2)\ldots$ Inverting both sides gives: $1/P = [1/(1 - 1/p_1^2)][1/(1 - 1/p_2^2)][1/(1 - 1/p_3^2)]\ldots$ But, $1/(1 - 1/p_1^2) = 1 + 1/p_1^2 + 1/p_1^4 + 1/p_1^6 + \cdots$, so $1/P = (1 + 1/p_1^2 + 1/p_1^4 + 1/p_1^6 + \ldots)(1 + 1/p_2^2 + 1/p_2^4 + 1/p_2^6 + \ldots)(1 + 1/p_3^2 + 1/p_3^4 + 1/p_3^6 + \cdots)\ldots = (1 + 1/2^2 + 1/2^4 + 1/2^6 + \ldots)(1 + 1/3^2 + 1/3^4 + 1/3^6 + \cdots)(1 + 1/5^2 + 1/5^4 + 1/5^6 + \ldots)\ldots$ Multiplying this out gives $1/P = 1 + 1/2^2 + 1/3^2 + 1/4^2 + 1/5^2 + 1/6^2 + \cdots$ which is just the sum of the reciprocals of the squares of the positive integers that Euler showed was equal to $\pi^2/6$. Therefore, $P = 6/\pi^2 = 0.60793$.

5 $\boldsymbol{V_N = 2.5S(1.05^{N-1} - 1.01^{N-1})}$ **and** $\boldsymbol{I_N = S[1.25(1.05)^{N-1} - 0.25(1.01)^{N-1}]}$**.** Let V_N, S_N, and I_N be the investment at the beginning of year N, the salary for year N, and the income for year N, respectively, with $V_1 = 0$ and $S_1 = S$. Salary increases 5% a year, so $S_N = S(1.05)^{N-1}$. Income is salary plus 10% of investment, so $I_N = S_N + 0.1V_N$. The total investment for a subsequent year increases from the previous year by 10% of the previous year's income: $V_N = V_{N-1} + 0.1(S_{N-1} + 0.1V_{N-1})$. Then, $V_N = 1.01V_N + 0.1S(1.05^{N-2}) = 0.1S\sum_0^{N-2} 1.01^i(1.05^{N-2-i}) = 0.1S(1.05^{N-2})\sum_0^{N-2}(1.01/1.05)^i = 0.1S(1.05^{N-2})[(1 - (1.01/1.05)^{N-1})/(1-1.01/1.05)] = 0.1S(1/1.05)(1.05^{N-1}-1.01^{N-1})/(1 - 1.01/1.05) = 0.1S(1.05^{N-1} - 1.01^{N-1})/(1.05 - 1.01) = 2.5S(1.05^{N-1} - 1.01^{N-1})$. Then, $I_N = 1.05^{N-1}S + 0.1S(2.5)(1.05^{N-1} - 1.01^{N-1}) = S[1.25(1.05^{N-1}) - 0.25(1.01^{N-1})]$

Bonus $\boldsymbol{a = 41{,}472(180^{12})}$**,** $\boldsymbol{b = 1{,}728(180^8)}$**,** $\boldsymbol{c = 288(180^6)}$**, and** $\boldsymbol{d = 288(180^7) =}$ **1,763,193,692,160,000,000.** Let $a = AF^{12}$, $b = BF^8$, $c = CF^6$ and $d = DF^7$. Substituting into the given equations, we have $A^2 + B^3 = C^4$ and $A^4 + B^6 = D^7F$. Then, find a triple (A, B, C) that solves the first equation. For example, $A = 41{,}472 = 2^9 3^4$, $B = 1{,}728 = 2^6 3^3$ and $C = 288 = 2^5 3^2$ is one such triple. Substituting these values into the second equation gives $A^4 + B^6 = 41{,}472^4 + 1{,}728^6 = 2^{36}3^{16} + 2^{36}3^{18} = 2^{36}3^{16}(1 + 3^2) = 2^{36}3^{16}10 = (2^5 3^2)^7 2^2 3^2 5$, where we have solved for the largest D by factoring out powers of 7; this makes $F = 2^2 3^2 5 = 180$. So, $D = 2^5 3^2 = 288$. The values of a, b, c and d follow. The above selected values for A, B, and C result in the smallest d that we have found.

Computer Bonus You would expect to toss the die about

44.25 times to complete drawing the beetle, or, to be exact, 44 + 1,006,625/3,981,312 tosses. The problem can be modeled as a large finite state machine. A "state" is a particular (allowed) assemblage of the beetle's body parts. Order of addition is not relevant, so tossing body-head-leg results in the same state as tossing body-leg-head. To win a game, you start with nothing and end with a complete beetle. On each toss of the die, either the state stays the same or one body part is added, transitioning the beetle into a new state. (After 13 tosses, the beetle assumes one of 141 possible states, ranging from zero parts to a complete beetle consisting of a body, head, tail, 6 legs, 2 eyes, and 2 antennae.) The problem asks for the expected value (EV) of the number of tosses required to reach the state of a complete beetle. For a given state, the EV equals the average EV's of the states that transition to it plus the expected number of tosses to transition out of it. (The expected number of tosses to transition out of a state equals 6/number of unique ways to leave that state.) One approach is to work backwards from a complete beetle to nothing to determine the states. Doing this results in 141 simple equations. The judges wrote a program that calculates the probability of each of the 141 states after N tosses and found that the EV of a completed beetle converged to a repeating decimal equal to the ratio 176,184,353/3,981,312. Alternatively, one can write a Monte Carlo program and get a close approximation to the above answer.

NEW WINTER PROBLEMS

1 I have found six different six-digit positive integers, whose sum also has six digits, with all 42 digits having one of only two different values. If I told you the sum, you would be able to identify all six numbers (no leading zeros). What is this sum?

—An Enigma by Jan Kay in *New Scientist*

2 You are to paint a totem pole, consisting of twelve animal images stacked on top of each other, using only two colors, blue and yellow. No two adjacent animals can be painted blue, although consecutive yellow colors are allowed. In how many different ways can you paint the totem pole?

—*BrainMatics Logic Puzzles* by Ivan Moscovich

3 Solve the following two cryptic addition problems simultaneously, that is, letters have the same values in both cryptics. Each different letter stands for a different digit. The same letter always stands for the same digit. There are no leading zeros. An * can stand for any digit.

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******        *SEVEN**
******        *SEVEN**
FOUR**        FOURTEEN
SEVEN*
ELEVEN
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What is the value of FOURTEEN?

—An Enigma by Albert Haddad in *New Scientist*

4 "CRYPTOQUIP" is a puzzle in a lot of newspapers. It consists of a coded sentence or two with each letter of the message represented by a different unique letter of the alphabet. If all 26 letters are used, how many codes are possible? Either a closed form equation or an approximate numerical value is a permissible answer.

—**Don A. Dechman,** *TX A '57*

5 Consider a five-by-five array of 25 points, one cm apart horizontally and vertically. Select five different points, none of which lie on either main diagonal of the array, such that the distances between pairs of points are all different. There are multiple solutions, but you only need to furnish one. Consider the array as consisting of five rows by five columns, each numbered 1 to 5, and express your answer as five pairs of numbers giving the row and column of the selected points.

—*Puzzles 101: A PuzzleMaster's Challenge* by Nobuyuki Yoshigahara

Bonus Given a three-by-three grid of pigeonholes, fastened to a wall with the middle hole covered, place envelopes in any or all of the outside holes so that the sum of the number of envelopes on each of the four sides has the same value, *N*. For example, if *N* is 10, one solution is shown in the accompanying figure. How many ways, including reversals, rotations, and reflections as different arrangements, can this be done? Express your answer as a function of *N*.

1	9	0
7		8
2	6	2

—*The Canterbury Puzzles* by H.E. Dudeney

Computer Bonus. Find two different ten-digit numbers, each using the digits 0 through 9 exactly once, such that their square roots are the reverses of each other.

—*You'd Better Be Really Smart Brain Bafflers* by Tim Sole and Rod Marshall

Send your answers to any or all of the Winter Brain Ticklers to **Curt Gomulinski, Tau Beta Pi, P.O. Box 2697, Knoxville, TN 37901-2697** or email to *BrainTicklers@tbp.org* as plain text only. The cutoff date for entries to the Winter column is the appearance of the Spring *Bent* in early April. The method of solution is not necessary. We welcome any interesting problems that might be suitable for the column. The Computer Bonus is not graded. Curt will forward your entries to the judges who are **H.G. McIlvried III,** *PA Γ '53;* **F.J. Tydeman,** *CA Δ '73;* **J.C. Rasbold,** *OH A '83;* and the columnist for this issue: **D.A. Dechman,** *TX A '57.*