Brain Ticklers

RESULTS FROM SUMMER 2012

Perfect			
Bohdan, Timothy E.	IN	Γ	’85
Gaston, Charles A.	PA	Β	’61
Jordan, Byron L.	MS	Α	’73
*Norris, Thomas G.	OK	Α	’56
Norris, Thomas G., Jr.	PA	Γ	’79
Prince, Lawrence R.	CT	Β	’91
*Schmidt, V. Hugo	WA	Β	’51
Slegel, Timothy J.	PA	Α	’80
*Spong, Robert N.	UT	Α	’58
*Stribling, Jeffrey R.	CA	Α	’92
*Thaller, David B.	MA	Β	’93
*Voellinger, Edward J.	Non-member		
Other			
Aron, Gert	IA	Β	’58
Bertrand, Richard M.	WI	Β	’73
Brule, John D.	MI	Β	’49
Couillard, J. Gregory	IL	Α	’89
Cross, Harry L.	MI	Γ	’65
Eckley, Paul L.	NV	Α	’75
Gerken, Gary M.	CA	Η	’86
Glaze, James B., Jr.	KS	Γ	’75
Golembiewski, Steven L.	PA	Β	’90
Handley, Vernon K.	GA	Α	’86
Haney, William S., Jr.	MD	Β	’57
Havas, Donald W.	NY	Ν	’67
Hengesbach, John P.	MI	Δ	’72
James, Catherine A.	Wife of member		
Jones, Donlan F.	CA	Ζ	’52
Jones, John F.	WI	Α	’59
Jones, Jeffery C.	Son of member		
*Kramer, J. David R.	PA	Δ	’57
Lalinsky, Mark A.	MI	Γ	’77
Marks, Lawrence B.	NY	Ι	’81
Marrone, James D.	IN	Α	’87
McKee, Eric P.Z.	Son of member		
Partanen, Thomas A.	MI	Β	’70
Peters, Christopher J.	MO	Γ	’09
Pineault, Wayne	IL	Γ	’79
Quintana, Juan S.	OH	Θ	’62
Rasbold, J. Charles	OH	Α	’83
Shah, Parth	Son of member		
*Strong, Michael D.	PA	Α	’84
Svetlik, J. Frank	MI	Α	’67
Walk, Daniel B.	TX	Α	’11
*Whitman, Alan M.	PA	Δ	’58

*Denotes correct bonus solution

SUMMER REVIEW

Problems 4 (shared birthdays) and 5 (knights on chessboard) were the ones that tripped up most people. Less than half of the Bonus (falling rod) answers submitted were correct.

FALL SOLUTIONS

Readers’ entries for the Fall problems will be acknowledged in the Spring ’13 BENT. Meanwhile, here are the answers:

1 The initial velocities of the 2 kg and 26 kg rockets are 18 m/s and 5 m/s, respectively. After the short bursts of each engine, the new velocities are 649 m/s and 180 m/s, respectively. Kinetic energy (Joules) equals $mv^2/2$, where m is mass (kilograms) and v is velocity (meters per second). Letting v = velocity of the 2 kg rocket and V = velocity of the 26 kg rocket, we have $2v^2/2 - 26V^2/2 = v^2 - 13V^2 = \pm 1$, which is a Pell equation. Trying a few values for V quickly shows that V = 5 and v = 18 is a solution. Most books on number theory show that if you have the smallest solution (s, t) to a Pell equation, other solutions can be generated by $(s - t\sqrt{N})^n = S - T\sqrt{N}$, where (S, T) is a new solution. For our case, $(18 - 5\sqrt{13})^2 = 649 - 180\sqrt{13}$. So, after engine firing, the new velocities are 649 m/s and 180 m/s.

2 The six-digit number that meets all the required statements is 621279. Let ABCDEF represent the desired answer. Then, from the required statements, A = B + 4 and C = B – 1. So, try B = 1, 2, 3, 4, and 5, one at a time, and only B = 2 satisfies all the remaining statements.

3 A minimum of four weighings is required. Any of the nine bottles can be heavier than the rest, and for each of these, there are eight possibilities for the lighter bottle, or they could all weigh the same. Thus, there are 9(8) + 1 = 73 possibilities. Each weighing gives three pieces of information—pans balance, left pan heavy, or right pan heavy. To account for all cases, we must have $3^N \geq 73$; since $3^3 = 27$ and $3^4 = 81$, $N = 4$, that is, four weighings are needed. One set that works is to weigh: (a) 1, 2, 3—4, 5, 6; (b) 1, 5, 7—2, 4, 8; (c) 1, 4—2, 5; and (d) 3, 6—7, 8. Each possibility for the bottles will give a different result for the four weighings. For example, suppose 5 is heavy and 7 is light; then the result of the four weighings will be RBRL, where R means right pan heavy and L means left pan heavy. As another example, suppose 2 is light and 8 is heavy; then we get RBLR.

4 The probability that Alice and Beth throw the same number of heads when tossing n coins is $C(2n, n)/4^n$, where $C(n, m)$ is the number of combinations of n things taken m at a time. The number of ways for Alice to get m heads when tossing n coins is $C(n, m)$; and, since there are 2^n ways for n coins to land, the probability of m heads is $C(n, m)/2^n$, and the probability that both Alice and Beth throw m heads is $[C(n, m)/2^n]^2 = [C(n, m)]^2/4^n$. The total probability that Alice and Beth throw the same number of heads is $\sum^n_{m=0} [C(n, m)]^2/4^n = 1/2^{2n}\sum^n_{m=0} [C(n, m)]^2$. From a table of sums of series (see, for example, "Handbook of Mathematical Formulas and Integrals" by Alan Jeffrey), we find that the summation equals $C(2n, n)$, so the desired probability is $C(2n, n)/4^n$. Alternatively, one can start with 2 and calculate the probability for increasing values of n until the above pattern becomes clear.

5 The hose can wet 64.7 m² of the wall. Most physics books derive the following equation for the trajectory of an object traveling under the influence of gravity: $y = y_o + x\tan\theta - gx^2/(2v^2\cos^2\theta)$, where y = vertical distance, x = horizontal distance, θ = vertical angle, g = acceleration due to gravity, and v = initial velocity. Let x_o = closest distance of nozzle to the wall, and x = horizontal distance from nozzle to a point on the wall. To achieve maximum wetting, we want maximum y at a given x. Since $1/\cos\theta = \sec\theta$, we have $y = y_o + x\tan\theta - (gx^2/2v^2)\sec^2\theta$, which upon differentiating with respect to θ, and setting equal to 0 gives $dy/d\theta = 0 = x\sec^2\theta - (gx^2/2v^2)(2\sec^2\theta\tan\theta)$. Solving for $\tan\theta$ gives $\tan\theta = v^2/gx$. Substituting this result into our initial equation and making use of the identity $\sec^2\theta = \tan^2\theta + 1$, gives $y = y_o + v^2/2g - gx^2/2v^2$. Maximum x occurs when the stream of water just hits the base of the wall, i.e., when $y = 0$. Setting $y = 0$ and solving for x gives $x_m = (v/g)(2gy_o + v^2)^{0.5}$. The wetted area A is twice the integral of ydz between the limits of z_m and 0, where z is the distance along the wall starting at the point directly in front of the nozzle, and z_m is the maximum value of z. We make use of the fact that $z^2 + x_o^2 = x^2$ and observe that z_m occurs at the same point as x_m. Thus, $z_m = (v/g)(2gy_o + v^2 - g^2x_o^2/v^2)^{0.5}$. Now, substituting for x^2 in the equation for y, we get $A = 2\int_0^{zm}(y_0 + v^2/2g - gx_o^2/2v^2 - gz^2/2v^2)dz = 2[(y_0 + v^2/2g - gx_o^2/2v^2)z_m - gz_m^3/6v^2]_0^{zm} = (2^{2.5}v/3g^{0.5})(y_0 + v^2/2g - gx_o^2/2v^2)$. Using the values $y_o = 1$ m, $v = 10$ m/s, $x_o = 5$ m, and $g = 9.81$ m/s^2, we get $z_m = 9.96486$ and $A = 64.7$ m^2.

Bonus The exact latitude where the plane crosses the starting longitude for the first time is $2\arctan(e^{2\pi}) - \pi/2$ radians or approximately 89° 47' 10". Consider a set of spherical coordinates with origin at the center of the Earth. Let θ = east longitude and α = north latitude. Let dx = distance traveled east when θ increases by $d\theta$, and dy = distance traveled north when α increases by $d\alpha$. Then, $dy = Rd\alpha$, where R is the radius of the Earth, and $dx = R\cos\alpha d\theta$. (The $\cos\alpha$ term arises because the distance between lines of longitude decreases as you move toward the pole.) Since the plane is on a constant northeast bearing, $dx = dy$, so $R\cos\alpha d\theta = Rd\alpha$, or rearranging, $d\theta = d\alpha/\cos\alpha$. Integrating both sides and assuming the plane starts at $\theta = 0$ and $\alpha = 0$ gives $\theta = \int_0^\alpha d\alpha/\cos\alpha = \ln[\tan(\pi/4 + \alpha/2)]$. We want α when $\theta = 2\pi$, that is $\ln[\tan(\pi/4 + \alpha/2)] = 2\pi$. Taking exponentials gives $\tan(\pi/4 + \alpha/2) = e^{2\pi}$. Solving for α gives $\alpha = 2\arctan(e^{2\pi}) - \pi/2 = 1.5670614457$ radians = 89.7860070747° = 89° 47' 10".

Computer Bonus The solution for J · FROULA = RETIRE · S with maximum REST, in base 12, is B(1A4596) = (A307A3)2 where A = 10 and B = 11. A simple computer program quickly determines that, of the six possible answers, this gives the maximum REST.

NEW WINTER PROBLEMS

1 I am participating in a Super Bowl XLVII betting pool. You know, the kind consisting of a 10x10 grid with the cells numbered 00 to 99, where the first digit refers to the last digit of the AFL team's score, and the second digit refers to the last digit of the NFL team's score. Each participant pays $1 per square, and $25 is paid to the person holding the winning number at the end of each quarter. I paid $2 for two squares, picked two slips at random from the basket, and got 00 and 88. A friend who arrived after all the squares were taken, wants to buy my numbers. What is a fair price for each of my numbers? The following table provides data on the final digits for the quarterly scores of the 46 previous Super Bowls.

Final Digit	0	1	2	3	4	5	6	7	8	9
Number of Appearances	101	21	9	56	38	10	30	77	9	17

—**Don A. Dechman,** *TX A '57*

2 Two spherical soap bubbles, each 10 cm in diameter, coalesce to form a "double bubble" consisting of two truncated spheres joined at their flat surfaces with a plane circular interface between them. What is the area of the interface? Assume that the volume of the double bubble is equal to the sum of the volumes of the two original bubbles.

—**Howard G. McIlvried III,** *PA Γ '53*

3 The newly crowned king of Cashtopia has decided to replace the coins in use with a new coinage bearing his image. Being interested in efficiency, he wants to have only three different coins. Furthermore, he wants the denominations chosen so that the average number of coins required to make change for any amount from 1 cent through 99 cents is a minimum. Assuming that the need to make change for any value between 1 and 99 is equally likely, what three denominations should he choose? If more than one set gives the minimum, the king prefers the one with the smallest sum for the three coins.

—*Doctor Ecco's Cyberpuzzles* by **Dennis E. Shasha,** *CT A '77*

4 How many distinct trapezoidal decompositions (sum of consecutive positive integers) does n have? For example, 15 = 7+8 = 4+5+6 = 1+2+3+4+5 has three.

—*Technology Review*

5 The game of Spot-It consists of a deck of cards, with eight pictures on each card. Between any two cards, there is one, and only one, matching picture. If each picture occurs the same number of times in the full deck, what is the maximum number of cards in the deck?

—Steve Schaefer via *mathrec.org*

Bonus N&Ns come in two colors—red and green—and there is always the same total number in each package (less than 50) with the same ratio of red to green. Al and Beth have each eaten two packages of N&Ns over the last couple of days. Al just reaches into the package and randomly chooses an N&N. With his first package, when he had eaten the last red N&N, he saw that he had one green left. With his second package, when he had eaten the last red, he had two greens left. He later calculated that the situation with the first package is four times as likely as the situation with the second package. Beth uses a different approach; until she runs out of one color, she flips a coin and eats a red if heads and a green if tails. With her first package, when she had eaten the last red N&N, she noted the number of greens left. With her second package, when she had eaten the last red, she had one more green left than with her first package. Beth also made some calculations and determined that the number of greens left with the first package is four times as likely as the number left with the second package. How many red and how many green N&Ns are in a package?

—Susan Denham in *New Scientist*

Double Bonus. What is the remainder when $2^{4,700,063,497}$ is divided by 4,700,063,497?

—Adapted from *Prime Numbers, the Most Mysterious Figures in Math* by David Wells

Send your answers to any or all of the Winter Brain Ticklers to: **Curt Gomulinski, Tau Beta Pi, P. O. Box 2697, Knoxville, TN 37901-2697** or email to *BrainTicklers@tbp.org* only as plain text. The cutoff date for entries to the Winter column is the appearance of the Spring BENT in early April. The method of solution is not necessary. We welcome any interesting problems that might be suitable for the column. The Double Bonus is not graded. Curt will forward your entries to the judges who are **H. G. McIlvried III,** *PA Γ '53;* **F.J. Tydeman,** *CA Δ '73;* **J.L. Bradshaw,** *PA A '82;* and the columnist for this issue,

—**D. A. Dechman,** *TX A '57.*