Brain Ticklers

RESULTS FROM SUMMER 2011

Perfect	
Brule, John D.	MI B '49
Couillard, J. Gregory	IL A '89
Gerken, Gary M.	CA H '11
Rasbold, J. Charles	OH A '83
Spong, Robert N.	UT A '58
*Thaller, David B.	MA B '93
Other	
Aron, Gert	IA B '58
Beaudet, Paul R.	Father of member
Bertrand, Richard M.	WI B '73
Conway, David B.	TX I '79
Davis, John H.	OH G '60
Dror-Rein, Elana	CA E '83
Rein, Edan	Son of member
Jones, Donlan F.	CA Z '52
Jones, Robert F.	Son of member
Jones, Michael P.	Son of member
Jones, John F.	WI A '59
Jones, Jeffery C.	Son of member
Kimsey, David B.	AL A '71
Lalinsky, Mark A.	MI G '77
Mitchell, Teri L.	AL A '85
Mitchell, Steven D.	AL A '12
Prince, Lawrence R.	CT B '91
Rentz, Peter E.	IN A '55
Schmidt, V. Hugo	WA B '51
Shah, Parth	Son of member
Stribling, Jeffrey R.	CA A '92
Strong, Michael D.	PA A '84
Tsiatis, Anastasius A.	Husband of member
Voellinger, Edward J.	Non-member
Zwillenberg, Melvin L.	NY I '60

* Denotes correct bonus solution

SUMMER REVIEW

Of the regular Summer problems, number 3 about the poker hand proved to be the most difficult. The Bonus about folding a map was unusually hard, as we received only one correct solution.

FALL SOLUTIONS

Reader's entries for the Fall problems will be acknowledged in the Spring Bent. Meanwhile, here are the answers:

1 The bill was $367.92 and the 72 turkeys cost $5.11 each. With a handheld calculator, divide 167.95 through 967.95 by 72 and note that 367.95 yields a unit cost closest to an integral number of cents and that 72 divides 367.92 exactly.

2 The equation of the parabola that passes through the four given data points is $y^2 + 3x^2 \pm (2\sqrt{3})xy - 2y - 3 = 0$. The generalized equation for a parabola is $ax^2 + bxy + cy^2 + dx + ey + f = 0$, with the further condition that $b^2 = 4ac$, where a and c aren't both zero. (Wikipedia on the internet is one source of these requirements.) Substituting data points (3,0) and (-1,0) in the generalized equation yields e = -2c. And, substituting (0,1) and (0,-1) yields d = 0 and a = -f. The generalized equation then reduces to $ax^2 + bxy + cy^2 - 2cy - a = 0$. Now, substituting data point (3,0) in this equation gives $9c - 6c - a = 0$ or $a = 3c$. Set c = 1 and a = 3 and the equation becomes $3x^2 + bxy + y^2 - 2y - 3 = 0$. The added constraint that for a parabola, $b^2 = 4ac = 4(3)(1)$ means that $b = \pm 2\sqrt{3}$. Thus, there are actually two parabolas that satisfy the given conditions, one inclined to the right and one inclined to the left.

3 When Beth says, "I know my number" at her eighth opportunity, she had just asked if Ann had a 15, received a "yes" answer, and knew she must have been assigned a 17. At Ann's first opportunity, the only question she can logically ask that will allow her to know her number is, "Is Beth's number a 1?" where a "yes" response would allow her to know she had a 1. Then, at Beth's first opportunity, the only question she can logically ask is, "Is Ann's number 1?" where a "yes" response would allow her to know she had a 3. Then, Ann, at her second opportunity, with two "I don't know my number" responses so far, would logically ask, "Is Beth's number a 3?" where a "yes" response would allow her to know she had a 3. With each additional "I don't know my number," Ann and Beth will deduce what next positive odd number to choose to ask about, until Ann, at her eighth opportunity with all negative responses so far, would ask if Beth had a 15. The "no" response would result in Ann's saying "I don't know my number," and that would prompt Beth to ask if Ann had a 15. She must have received a "yes" response, because she announces she knows her number.

4 The probability that the quadratic equation $x^2 + bx + c = 0$ has complex roots is $2/(3q^{0.5})$ for $q \geq 4$, and $0.5 - q/24$ for $q \leq 4$, where b and c are both in the range $\pm q$. At $q = 4$, both equations give a 1/3 probability. A requirement for the quadratic equation to have complex roots is $c > b^2/4$. Prepare a set of Cartesian coordinates with c on the vertical axis and b on the horizontal axis, and plot the parabola $c = b^2/4$. Draw a square with sides $b = \pm q$ and $c = \pm q$. The region yielding complex roots is bounded below by the parabola and above and on the right by the square. Therefore, we have the integral $\int_0^z (q - b^2/4)db$ to evaluate. There are two cases to consider: $q < 4$ for which the parabola will intersect the right edge of the square and $q > 4$ for which the parabola will intersect the top of the square, so the two cases require different upper limits for the integral. For the first case, $z = q$. Using Calculus 101, the value of the integral is $q^2 - q^3/12$, which must be multiplied by 2 (because b can be either positive or negative) and divided by $4q^2$, the total variable space (area of the square). Therefore, $P_{q<4} = 0.5 - q/24$. For the second case ($q > 4$), $z = 2q^{0.5}$ which gives a value of the integral of $4q^{1.5}/3$ and $P_{q>4} = 2/(3q^{0.5})$.

5 The pool-table cushion height should be $0.7D$ to create a rebound that maintains rolling-without-slipping motion. Conceptually, there is no difference between hitting a stationary ball with a moving cue and hitting a stationary cushion with a moving ball. Therefore, what we are looking for is the "sweet spot," the point on the ball that, when hit, results in rolling-without-slipping motion. From Newton's second law, $F_x = ma_G$, where F_x is the horizontal component of the force exerted by the cushion, m is the ball's mass, and a_G is the acceleration of the center of gravity of the ball. Let h equal the height of the cushion. Then, $F_x(h - r) = I\alpha$, where r is the ball's radius, I its moment of inertia, and α the angular acceleration. For a sphere, $I = 2mr^2/5$. Also, if the ball rolls without slipping, then $\alpha = a_G/r$. Substituting into the torque equation gives $ma_G(h - r) = (2mr^2/5)(a_G/r)$ or $h - r = 2r/5$, so $h = 7r/5 = 7D/10$.

Bonus The range of eccentricities that have an apogee greater than R_s is 0.591 to 0.755. According to Kepler's Law of Periods, $T^2 = (4\pi^2/GM)a^3$, where T is the period, G is the gravitational constant, M is the mass of the Earth, and a is the semi-major axis of the elliptical orbit. Once T is set, then a can be calculated, and all orbits with that T will have the same major axis. Substituting T = 12(3,600) = 43,200 s, G = 6.67 x 10^{-11} m^3/kg s^2, and M = 5.98x10^{24} kg into $a = (GMT^2/4\pi^2)^{1/3}$ gives a = 2.66x10^7 m. There are two limits for the orbit: (1) the perigee is 100 miles (plus the radius of the Earth) and (2) the apogee is R_s, the radius of the geosynchronous orbit. For Case 1, R_p = 100(1,610) + 6.37x10^6 = 6.531x10^6 m, and $R_a = 2a - R_p$ = 4.67x10^7 m. Now ea is defined as half the distance between the foci of the ellipse, so $R_p = a - ea$ and $R_a = a + ea$. Solving gives $e = (R_a - R_p)/2a$. For Case 1, this gives e_1 = (4.67x10^7 – 0.653x10^7)/(2x2.66x10^7) = 0.755. For Case 2, R_a = 26,300(1,610) = 4.23x10^7 m and $R_p = 2a - R_a$ = 1.09x10^7, which give e_2 = (4.23x10^7 – 1.09x10^7)/(2x2.66x10^7) = 0.591.

Double Bonus. The last seven digits of $7{,}777{,}777^{7{,}777{,}777}$ are 5,347,697. The general approach for solving problems of this type (i.e., what are the final n digits of N^E) is as follows. Create a table with four columns. In Column 1, write 0, 1, 2, 3, … In Column 2, enter the corresponding powers of 2 which are less than E, i.e., enter 1, 2, 4, 8, 16, … In Column 3, enter the last n digits of the powers of N corresponding to Column 2. This is easily done by starting with N and squaring each successive entry and retaining only the last n digits. Now, with the aid of Column 2, it is easy to express E as the sum of powers of 2. Next, transfer to Column 4 those entries in Column 3 that correspond to the powers of 2 that sum to E. Finally, starting at the top of Column 4, successively multiply each value by the next value, retaining only the last n digits; the final calculation will be the desired answer. For this problem, N = 7,777,777 and E = 7,777,777 = $2^{22} + 2^{21} + 2^{20} + 2^{18} + 2^{17} + 2^{15} + 2^{13} + 2^{11} + 2^{10} + 2^8 + 2^7 + 2^6 + 2^5 + 2^4 + 2^0$. Therefore, after creating Columns 1, 2, and 3, use Column 3 entries 1, 5, 6, 7, 8, 9, 11, 12, 14, 16, 18, 21, 22, and 23 to calculate the last 7 digits of $7{,}777{,}777^{7{,}777{,}777}$ as 5,347,697. The calculations are easily carried out on a spreadsheet, provided it has 16-digit precision.

NEW WINTER PROBLEMS

1 How many primes appear in the following sequence: 9; 98; 987; 9,876; …; 9,876,543,210; 98,765,432,109; 987,654,321,098; and so on?

—Daryl Cooper

2 Find two three-digit integers that sum to a four-digit integer such that each of the digits 0 through 9 is used, with no leading zeros. What are the maximum and minimum values for the four-digit integer?

—*The Playful Brain* by Richard Restak and Scott Kim

3 Bill has a problem. He has locked his Tau Beta Pi pin in a strong box and forgotten the combination. The lock uses all the digits 1 through 6 in some order. He has tried the following three combinations: 4-5-6-1-3-2 which has one digit correct; 6-2-3-4-5-1 which has two digits correct; and 2-3-1-6-4-5 which has three digits correct. What is the combination?

—*The Everything Brain Strain Book* by Jake Olefsky

4 Mac is idly tossing two dice when he decides to see how many tosses it would take to go from 1 to 6 in order. The rules are to toss the two dice until one or the other or both show a 1. Then, toss until a 2 shows. However, if both a 1 and a 2 show at the same time, both can be used. He then tosses for a 3 and similarly for other numbers. What is the expected number of tosses to go from 1 to 6? Once you have the answer, get a couple of dice and play the game a few times to see how closely the average matches your answer.

—**Howard G. McIlvried III**, *PA Γ '53*

5 A distinct positive integer has been assigned to each letter of the alphabet such that the letters of PLUTO add to 40; URANUS 36; NEPTUNE 29; SATURN 33; JUPITER 50; MARS 32; Earth 31; MOON 36; VENUS 39; MERCURY 33; and SUN 18. What is the value of PLANETS? Get a pencil and paper and solve this one without computer help!

—C. W. Haigh in *New Scientist*

Bonus. Start with a checker in the center of a 5x5 checkerboard. The objective is to move the checker from square to square so that each square is visited exactly once, except the center square to which you must return on your 25th move. The only legal moves are to a square that is two squares away diagonally or to a square that is three squares away horizontally or vertically. How many unique paths can the checker trace? Rotations, reflections, path reversals, and combinations thereof, are not considered different paths. This problem can be solved without computer help.

—**William W. Verkuilen,** *WI B '92*

Computer Bonus. Beginning with any multi-digit number, N, start a sequence by writing its digits in order, starting with the left-most digit. The next number in the sequence is the sum of the digits of N. Continue with a Fibonacci-like process, where the next term in the series is the sum of the previous n terms, where n is the number of digits in N. If N appears in this sequence, then N is a Keith number, named after Michael Keith, who discovered them. For example, 197 is a Keith number because 1, 9, 7, 17, 33, 57, 107, 197, … contains N. The four smallest Keith numbers that are also prime numbers are 19, 47, 61, and 197. What is the next smallest prime Keith number?

—**Fred J. Tydeman,** *CA Δ '73*

Send answers to any or all of these Brain Ticklers to **Curt Gomulinski, Tau Beta Pi, P. O. Box 2697, Knoxville, TN 37901-2697** or email to *BrainTicklers@tbp.org* only as plain text. The cutoff date for entries to the Winter column is the appearance of the Spring Bent in early April. The method of solution is not necessary. We welcome problems that might be suitable for the column. The Computer Bonus is not graded. Curt will forward entries to the judges who are **H. G. McIlvried III,** *PA Γ '53;* **F. J. Tydeman,** *CA Δ '73;* **J. L. Bradshaw,** *PA A '82;* and the columnist for this issue:

—**D. A. Dechman,** *TX A '57.*