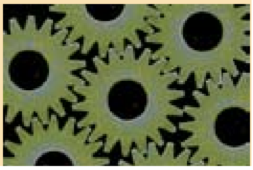


Brain Ticklers

RESULTS FROM SUMMER 2006

Perfect

*Baines, Elliot "Chip" A., Jr.	NY	Λ '78
*Bohdan, Timothy E.	IN	Γ '85
*De Vincentis, Joseph W.	TX	Γ '93
*Gerken, Gary	Non-member	
*Schmidt, V. Hugo	WA	Β '51
*Strong, Michael D.	PA	Α '84

Other

Alagappan, Vairam	IL	Α '86
Alexander, Jay A.	IL	Γ '86
Aron, Gert	IA	Β '58
Brady, Dawn	IL	Ε '05
Chatcavage, Edward F.	PA	Β '80
Craven, Corbin	Non-member	
Creutz, Michael J.	CA	Β '66
Creutz, Edward C.	PA	Γ '36
Doolittle, Scott	CA	ΑΑ '07
Gresho, Philip M.	IL	Α '61
Handley, Vernon K.	GA	Α '86
Hollingsworth, David	CA	Ρ '83
Jones, Donlan F.	CA	Ζ '52
Kern, Peter L.	NY	Λ '62
Mazeika, Daniel F.	PA	Β '55
*McClendon, Ryan R.	CT	Α '04
*Meerscheidt, Kyle	Husband of member	
Miller, Richard "Dick"	MA	Ζ '64
Quintana, Juan S.	OH	Θ '62
Rasbold, J. Charles	OH	Α '83
Rentz, Peter E.	IN	Α '55
Scholz, Greg	PA	Β '00
Spong, Robert N.	UT	Α '58
Svetlik, Frank	MI	Α '67
Tam, Helena L.	NY	Ι '01
Valko, Andrew G.	PA	Λ '80
Voellinger, Edward J.	Non-member	

* denotes correct bonus solution

SUMMER REVIEW

Joseph D. North, *FL A '04*, has pointed out that, in our statement of the Spring '06 Double Bonus, we should have said that the average Julian calendar year was longer than the *tropical* year, not the *sidereal* year.

FALL SOLUTIONS

Reader's entries for the Fall problems will be acknowledged in the Spring BENT. Meanwhile, here are the answers:

1 The lineup is Jane, Benny, dill; Jenny, Ray, hay; Jill, Bill, grain; Valerie, Charlie, celery; and Carly, Wayne, barley. Jane married Benny. Jenny married Ray. So Carly is not Ray's wife and must be Charlie's sister and grow barley. Jill and Bill are the only possible rhyming pair remaining, since Carly can't be Charlie's wife. So, Carly married Wayne and Valerie married Charlie. Jenny/Ray grew hay. Valerie/Charlie grew celery. Carly/Wayne grew barley. Jill/Bill didn't grow dill and thus grew grain. So Jane/Benny grew dill.

2 Each dog got $158. The possible range for 581 dogs to divide a five-digit estate is $17 to $172 per dog for a total of $10,458 to $99,932. Using a hand-held calculator or a spreadsheet, start with 10,458 and sequentially add 581 and you will find that only 91,798 satisfies the conditions that, if the estate were ABCDE, that AB + C = DE.

3 The rug's outer edge is 12 cm from the center of the cylinder. The length of the carpet's first course is 10π. The rug rises as it nears the starting edge of the carpet, but this adds only a small length with each course and doesn't affect the answer. The length of the second course is 12π. And, the length of six courses is (10 + 12 + 14 + 16 + 18 + 20)π = 282.7 cm. So the length of the seventh partial course is a bit less than 17.3 cm. The outer edge is 5 + 7 =12 cm from the center of the cylinder.

4 The expected distance from the nearest plum to the surface of the spherical plum pudding is $R/(3N+1)$. When N infinitely small plums are distributed at random within a sphere of radius R, the probability distribution for the plum with the largest distance from the center is:

$$p(r)dr = N\cdot\left(\frac{4\pi r^2 dr}{\frac{4}{3}\pi R^3}\right)\cdot\left(\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}\right)^{(N-1)}.$$

The expected radius, r, of the closest plum to the surface is

$$E(r) = \int_0^R r p(r)dr = 3NR/(3N+1).$$

The answer is the distance D from the sphere surface,

$$D = R - E(r) = R/(3N+1).$$

The first term in brackets on the right side of the equation for $p(r)dr$ is the probability of finding a plum within a spherical shell of radius $r + dr$. The second term in brackets is the probability that the remaining (N - 1) plums are found within a sphere of radius r. The N term accounts for the ordering in that the plum closest to the surface can be any one of the N plums.

5 The string can sweep out a maximum area of $\pi L^2/2 + L^3/(3R)$. This problem is a thinly disguised version of the more well known "The Bull and the Silo" problem. Traditional approaches to the area swept out by the string can become long and tedious. Michael E. Hoffman has developed a general solution (Amer. Math. Monthly 105, 55-58, (1998)) for the area swept out in a plane by a flexible string tied to a smooth, convex curve parameterized by its arc length $\vec{r}(s)$. The expression for the area swept out is

$$A = \frac{\pi L^2}{2} + \int_{-L}^{L}(L - |s|)^2\kappa(s)ds$$

where $\kappa(s) = \|\vec{r}''(s)\|$ is the curvature of the curve $\vec{r}(s)$. In the plane that the string sweeps, the curvature κ of the circle of radius R is $\kappa = 1/R$ and the area is

$$A = \frac{\pi L^2}{2} + \int_{-L}^{L}(L - s)^2\left(\frac{1}{R}\right)ds = \frac{\pi L^2}{2} + \frac{L^3}{3R}.$$

Bonus. The probability that the needle will cross a grid line is $[2L(d_1 + d_2) - L^2]/(\pi d_1 d_2)$. For two independent events, H and V, the probability that one or the other or both will occur is P(H) + P(V) – P(HV), where P(HV) is the probability that both H and V occur. This Tickler is a grid version of Buffon's needle problem, the parallel lines version of which has the well-known solution P = $2L/(\pi D)$, where

L is the needle's length and D is the distance between lines. If H is the needle's crossing a horizontal line and V is the needle's crossing a vertical line, then the desired probability is $P = 2L/(\pi d_1) + 2L/(\pi d_2) - P(HV)$. For the needle to cross both a horizontal and a vertical line, its center must be within a distance of $L/2$ of a grid intersection. Let $P(x, y)$ be a point within a quarter circle with center at an intersection and radius $L/2$. Draw two lines of length $L/2$ from P, one just touching a vertical line and one just touching a horizontal line. The probability that a needle whose center is at P will cross both lines is $(\alpha - \beta)/(\pi/2) = 2(\alpha - \beta)/\pi$, where α and β are the angles the two lines make with the horizontal. Since $\alpha = \cos^{-1}(2x/L)$ and $\beta = \sin^{-1}(2y/L)$, we have *P(HV)* equals:

$$\left(\frac{8}{\pi d_1 d_2}\right)\int_0^{\frac{L}{2}}\int_0^{\sqrt{\frac{L^2}{4}-x^2}}\left(\cos^{-1}\frac{2x}{L} - \sin^{-1}\frac{2y}{L}\right)dy\,dx =$$

$$\frac{2L^2}{\pi d_1 d_2}\int_0^{\frac{\pi}{2}}\int_0^{\alpha}(\alpha - \beta)\sin\alpha\cos\beta\, d\beta\, d\alpha$$

which equals $\frac{L^2}{\pi d_1 d_2}$. The second double integral arises from letting $x = (L\cos\alpha)/2$ and $y = (L\sin\beta)/2$. The factor $4/(d_1 d_2)$ occurs because there are four quarter circles in a grid square of area $d_1 d_2$, one at each intersection. The integration of the second double integral is straightforward and gives $L^2/(\pi d_1 d_2)$. Therefore, $P = 2L/(\pi d_1) + 2L/(\pi d_2) - L^2/(\pi d_1 d_2) = [2L(d_1 + d_2) - L^2]/(\pi d_1 d_2)$.

Double Bonus. The expected time interval for the monkeys to type a 24-character phrase, with one chance in 27 of success on each key stroke, is 7.14×10^{18} 365-day years. Quite a long time! It requires 27^{24} strikes, and the 10^8 monkeys have 10 strikes per second, and there are 31,536,000 seconds in a 365-day year. Do the math with a good hand-held calculator.

NEW WINTER PROBLEMS

1 A population of amoebae starts with one amoeba. If there is a 75 percent probability that an amoeba will survive and split to create two amoebae and a 25 percent probability that it will not survive and if all the amoebae in future generations have the same probability of survival, what is the probability that this population of amoebae will go on forever? All splits occur at the same time. And, at this time, each amoeba split or death is independent from the others' outcomes.

—*Classic Mathemagic* by R. Blum, A. Hart-Davis, B. Longe, and D. Niederman

2 What is the largest prime, p, less than 1,000 for which $N = p^3 + 2p^2 + p$ has exactly 42 positive integral factors? Remember that both 1 and N are factors of N.

—*Mathematics Teacher*

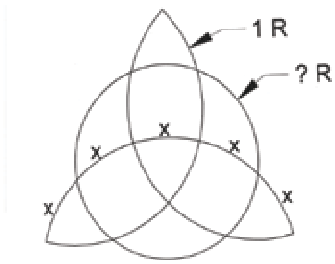
3 Solve the following cryptic multiplication problem. This can be solved without computer assistance.

LYNDON × B = JOHNSON

Each letter stands for a different digit, and there are no leading zeros.

—*The Numerology of Dr. Matrix* by Martin Gardner

4 On a popular television show, *Charmed*, the figure shown below appears on its *Book of Shadows*.



As seen, the figure has three intersecting arcs in a symmetrical pattern and an inner circle joining them together. The figure is constructed so that each of the five arcs marked x are of the same distance. If the radius of each of the three intersecting arcs is 1, then what is the radius of the inner circle? Express your answer accurate to four significant figures.

—**Dennis R. Dettman,** *NY N '77*

5 I have written down three different three-digit integers, no leading zeros, each of which contains at least one digit 3. For each number, three of the following statements are true, and three are false. (a) The number is prime. (b) The number is a cube of an integer. (c) The middle digit is the average of the other two digits. (d) The unit's digit differs from the ten's digit by 3. (e) The number has a two-digit prime factor, the digits of which differ by 3 or whose sum is a cube. (f) The number is a triangular number, that is, of the form $n(n + 1)/2$, which generates the sequence 1, 3, 6, 10, and so on. What are the three numbers?

—Adrian Somerfield in *New Scientist*

Bonus. Place seven points, each at a different position, on a plane so that, if you choose any three of them, at least two will be exactly 1 cm apart.

—*Classic Mathemagic* by R. Blum, A. Hart-Davis, B. Longe, and D. Niederman

Computer Bonus. For $TAU^{.BETA} = PI$, find TAU and BETA that give PI closest to an integer. Note the decimal point before BETA. Same rules as Problem No. 3. This one needs computer help!

—**Don A. Dechman,** *TX A '57*

Send your answers to any or all of the Winter Brain Ticklers to **Jim Froula, Tau Beta Pi, P. O. Box 2697, Knoxville, TN 37901-2697** or email as plain text to *BrainTicklers@tbp.org*. The cutoff date for entries to the Winter column is the appearance of the Spring BENT in early April. The method of solution is not necessary. We welcome any interesting problems that might be suitable for the column. The Computer Bonus is not graded. Jim will forward your entries to the judges who are **H. G. McIlvried III,** *PA Γ '53*; **F. J. Tydeman,** *CA Δ '73*; **J. L. Bradshaw,** *PA A '82*; and the columnist for this issue,

—**D. A. Dechman,** *TX A '57*.