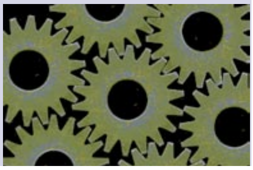


Brain Ticklers

RESULTS FROM SUMMER 2005

	Perfect			
*	Couillard, J. Gregory	IL	A	’89
	Mayer, Michael A.	IL	A	’89
*	Rasbold, J. Charles	OH	A	’83
*	Voellinger, Edward J.	non-member		
	Other			
*	Alexander, Jay A.	IL	Γ	’86
	Brule, John D.	MI	B	’49
*	Christenson, Ryan C.	UT	B	’93
*	Hess, Richard I.	CA	B	’62
	James, Cathy	wife of member		
	Jones, Donlan F.	CA	Z	’52
	Kassner, Rudy	non-member		
	Wagner, William R.	CA	E	’58
	Wagner, William C.	son of member		
	Lalinsky, Mark A.	MI	Γ	’77
*	Lonz, Gary J.	non-member		
	Mueller, Lisa	OH	K	’88
	Quintana, Juan S.	OH	Θ	’62
	Rasmussen, Dave	father of member		
	Rentz, Peter E.	IN	A	’55
	Taylor, Don	non-member		
	Schleehauf, Martin W.	NY	N	’79
*	Spong, Robert N.	UT	A	’58
*	Stribling, Jeffrey R.	CA	A	’92
	Valko, Andrew G.	PA	Λ	’80
	Van Houten, Karen	ID	A	’76
*	Vogt, Jack C.	OH	E	’56
	SPRING ENTRY-Perfect (rectification)			
	Carver, Robert M.	OH	K	’87

* denotes correct bonus solution

SUMMER REVIEW

Problem #2 on primes, problem #3 on anti-magic squares, and problem #4 on rational values were all equally hard, with about half the answers for each being correct.

Ray P. Steiner, *AZ* A *’63*, asked a math website if anyone could find eight integers such that any five would sum to a prime, and Wim Benthem found 71675, 94787, 145595, 168707, 219515, 242627, 316547, and 6151835. Congratulations, Wim!

FALL SOLUTIONS

Readers’ entries for the Fall problems will be acknowledged in the Spring BENT. Meanwhile, here are the answers:

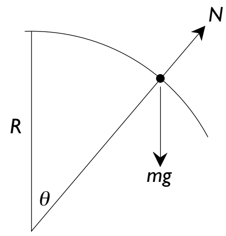
1 The solution for A/DE + B/FG + C/HJ = 1 is 5/34 + 7/68 + 9/12 = 1. Here it is easiest to write a simple computer program to check all of the possible combinations. If you would like a copy of his QBasic computer program, contact the editor at *dondechman@verizon.net.*

2 Rex tells the truth on Tuesdays. One at a time, assume he is telling the truth on each of the three days that he makes a statement, and also consider the case of his not telling the truth on any of the three statement days. The only case that works is Rex is telling the truth on the third day. On this third day, from the given statements, he lies on W and F; he tells the truth on M or Tu; and yesterday was not Th, Sa, or Su. So, he couldn’t be telling the truth on Monday because yesterday can’t be Sunday, and, thus, he must tell the truth on Tuesdays.

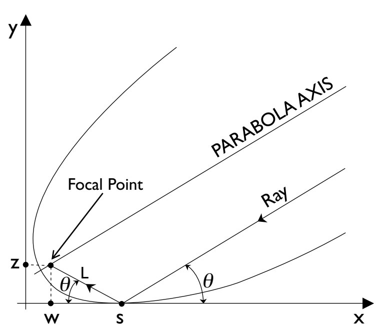
3 The dihedral angle between two faces of a regular tetrahedron is $2\arcsin(\sqrt{3}/3) \approx 70.529°$. Cut a vertical plane down through a unit regular tetrahedron, sitting on its triangular base, such that the plane cuts through one of the tetrahedron’s three slanted edges. The intersection of this plane with the tetrahedron is an isosceles triangle with a length of $\sqrt{3}/2$ for the base and one side and a length of 1 for the other side with the desired angle being opposite this other side. A little trigonometry quickly shows that this angle has a value of $2\arcsin(\sqrt{3}/3)$.

4 The contestants best pre-game strategy gives them a 75% chance of winning. Their best strategy is to agree that, if anyone sees two hats of the same color, he/she will guess that his/her hat is the opposite color. There are eight possible ways the hats can be distributed. Two of those are three red hats or three blue hats, in which case they lose. But the other six ways have two hats of one color and one hat of the other color, and for these cases they win.

5 The object leaves the hemisphere at a height of $2R/3$ above the table and impacts the table at a distance of $R(4\sqrt{23} + 5\sqrt{5})/27 \approx 1.1246R$ from the center of the hemisphere. The equation of motion for the object sliding frictionlessly down the hemisphere $d^2\theta/dt^2 = g(\sin\theta)/R$, where g is the gravitational constant and θ is defined in the figure below. The integral of this equation is $(d\theta/dt)^2 = 2g(1 - \cos\theta)/R$, which can be rewritten as $v^2 = 2gR(1 - \cos\theta)$, where $v = R\, d\theta/dt$ is the tangential velocity. In the radial direction, the force balance is $mg\cos\theta - N = mv^2/R$. As the object slides, v increases and N decreases until $N = 0$. At this point, $mv^2/R = mg\cos\theta$. Combining this equation with $v^2 = 2gR(1 - \cos\theta)$ yields $\cos\theta = 2/3$. So, the object leaves the surface at a height of $2R/3$, and at this point the problem becomes a fairly simple trajectory problem where the initial velocity, path angle, and height are known. One can then calculate that the object will impact the table at about $1.1246R$ from the center of the hemisphere.



Bonus The equation traced by the locus of the focus of the parabola, $y = ax^2$, as it rolls without slipping against the x-axis is the catenary $z = (e^{4aw} + e^{-4aw})/8a$, where (w, z) represents a point on the locus. Shown below is an example of a rolled parabola, touching the x-axis at point, S. One feature of a parabola is that all rays coming in parallel to the parabola’s axis reflect to the focal point. Consider a ray coming in



at an angle θ. If the parabola were rotated back to its original position, you would observe that θ is the compliment of the angle made by the tangent to the parabola at a distance S along its length. Therefore, $\tan\theta = dx/dy = 1/2ax$. So, $\sin\theta = 1/(4a^2x^2 + 1)^{1/2}$ and $\cos\theta = 2ax/(4a^2x^2 + 1)^{1/2}$. The focal point is at $(0, 1/4a)$ before the parabola is rolled. Using the formula for arc length and integrating from 0 to x yields $S = \ln[2ax + (4a^2x^2 + 1)^{1/2}]/4a + x(4a^2x^2 + 1)^{1/2}/2$ and $L = [(ax^2 - 1/4a)^2 + x^2]^{1/2} = (4a^2x^2 + 1)/4a$. Now, $w = S - L\cos\theta = \ln[2ax + (4a^2x^2 + 1)^{1/2}]/4a$. So, $e^{4aw} = 2ax + (4a^2x^2 + 1)^{1/2}$ which, on solving for x, gives $x = (e^{4aw} - e^{-4aw})/4a$. Now, $z = L\sin\theta = (4a^2x^2 + 1)^{1/2}/4a = e^{4aw}/4a - x/2 = (e^{4aw} + e^{-4aw})/8a = \cosh(4aw)/4a$.

Double Bonus The probability that a line joining two randomly chosen points on an infinite square lattice will not pass through another lattice point is $6/\pi^2 \approx 0.607927$. Let X and Y be the horizontal and vertical distances, respectively, between the two points. For the connecting line segment to pass through an intermediate lattice point, X and Y must have a common factor. A randomly chosen integer will be divisible by a given prime p with a probability of $1/p$ (for example, every fifth integer is divided by five). For two integers chosen at random, the probability that they do not share a common factor p is $1 - 1/p^2$. Finally, the probability P that two randomly chosen integers share no common factors is the product of $1 - 1/p^2$ for all primes. Thus, $1/P$ is the product of terms of the form $1/(1 - 1/p^2) = (1 + 1/p^2 + 1/p^4 + 1/p^6 + \dots)$ for all the primes. A little thought will show that $1/P$ is just the sum of the inverses of the squares of all the positive integers, which Euler showed is equal to $\pi^2/6$. Thus, $P = 6/\pi^2$. You can write a simple computer program that will quickly converge to about 0.607927.

NEW WINTER PROBLEMS

1 One hundred parishioners attended our church picnic, and 90 had a hot dog, 80 had potato salad, 70 had cake, and 60 had ice cream. However, no one ate all four items. How many had both cake and ice cream?

—*Classic Mathemagic* by R.A.Blum, A. Hart-Davis, B. Longe, and D. Niederman

2 Al, Sam, and Ralph play a challenge tennis tournament in which two of them play a set, with the winner staying on the court to play the person who sat out. In the course of the tournament, Al played 15 sets, Sam played 14 sets, and Ralph played 9 sets. Who played in set #13?

—**Richard I. Hess**, *CA B '62*

3 In the following equations, each letter always has the same value, but different letters may have the same value, and values may be positive, negative, zero, integer, or fractional. Given that U+N=1; D+E+U+X=2; T+R+O+I+S=3; Q+U+A+T+R+E=4; C+I+N+Q=5; S+I+X=6; S+E+P+T=7; H+U+I+T=8; N+E+U+F=9; D+I+X=10; O+N+Z+E=11; D+O+U+Z+E=12; T+R+E+I+Z+E=13; Q+U+A+T+O+R+Z+E=14; and Q+U+I+N+Z+E=15, what is the value of S+A+I+N+T+T+R+O+P+E+Z?

—C.W. Haigh in *New Scientist*

4 On my last birthday, I received a present packed in a rectangular box, 10 cm by 20 cm by 30 cm. The box was wrapped by an inelastic continuous loop of very narrow ribbon that crossed all six faces at least once, held in place with transparent tape, with the total length of ribbon on each face being at least 1 cm. What is the shortest ribbon that meets these requirements? Express your answer accurate to three decimal places.

—**James M. Garnett III**, *MS A'65*

5 Five friends, Andrew, Bernie, Cliff, Don, and Ed, each has a son and a daughter. Their families are so close that each friend's daughter is married to the son of one of his friends. As a result, the daughter-in-law of the father of Andrew's son-in-law is the sister-in-law of Bernie's daughter. And, the son-in-law of the father of Cliff's daughter-in-law is the brother-in-law of Don's son. Furthermore, the daughter-in-law of the father of Bernie's daughter-in-law has the same mother-in-law as the son-in-law of the father of Don's son-in-law, and the son-in-law of the father of Ed's son-in-law has the same father-in-law as the daughter-in-law of the father of Cliff's daughter-in-law. However, the situation is simplified by the fact that no daughter-in-law is the sister-in-law of the daughter of her father-in-law. Who is married to whom?

—*100 Games of Logic* by Pierre Berloquin

Bonus. A mathematics professor smokes a pipe. He carries two identical matchboxes, originally containing 20 matches each. When he lights his pipe, he selects a matchbox at random and lights his pipe with one match and discards the used match. There will come an occasion when he first selects a matchbox with only one match in it. At this point, what is the expected number of matches in the other box? Express your answer accurate to three decimal places.

—Adapted from *An Introduction to Probability Theory and Its Applications* by William Feller

Computer Bonus. Solve the following cryptic multiplication problem.

GEORGE = HW × BUSH

—**H.G. McIlvried III**, *PA Γ '53*

Send your answers to any or all of the Winter Brain Ticklers to: **Jim Froula, Tau Beta Pi, P. O. Box 2697, Knoxville, TN 37901-2697** or email to *BrainTicklers@tbp.org* only as plain text. The cutoff date for entries to the Winter column is the appearance of the Spring BENT in early April. The method of solution is not necessary. We welcome any interesting problems that might be suitable for the column. The Computer Bonus is not graded. Jim will forward your entries to the judges who are **H.G. McIlvried III,** *PA Γ '53*; **F.J. Tydeman**, *CA Δ '73*; **J.L. Bradshaw**, *PA A '82*; and the columnist for this issue,

—**D.A. Dechman**, *TX A '57*.