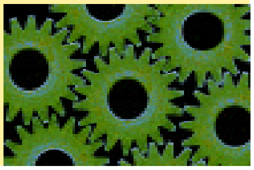


Brain Ticklers

RESULTS FROM SUMMER 2004

Perfect

Baines, "Chip" A., Jr.	NY	Δ	'78
* Barker, Alva Clifford	CA	Ε	'59
* Brana-Mulero, Francisco	PR	Α	'74
* Celestino, James R.	NJ	Β	'00
* Couillard, J. Gregory	IL	Α	'89
Creutz, Michael J.	CA	Β	'66
Creutz, Edward C.	PA	Γ	'36
* Doniger, Kenneth J.	CA	Α	'77
* Fuemmeler, Jason A.	OH	Θ	'00
Galer, Craig K.	MI	Α	'77
* Garnett, James M.	MS	Α	'65
* Goodman, Thomas J.	PA	Η	'05
* Griggs, James L., Jr.	OH	Α	'56
* Marx, Kenneth D.	OR	Α	'61
* Matusz, Robert J.	MI	Γ	'82
* Mayer, Michael A.	IL	Α	'89
* Nabutovsky, Joseph	Father of member		
* Peithman, Harlan W., Jr.	IL	Α	'53
* Post, Irving G.	PA	Ε	'58
* Powell, Scott F.	NC	Γ	'97
Quintana, Juan S.	OH	Θ	'62
* Schmidt, V. Hugo	WA	Β	'51
Snyder, M. Duane	IA	Β	'63
* Spong, Robert N.	UT	Α	'58
* Strong, Michael D.	PA	Α	'84
* Teale, John L.	NM	Β	'76
Franklin, Bonnie	Non-member		
* Thaller, David B.	MA	Β	'93
* VanShaar, Steven R.	UT	Γ	'00
* Wegener, Stephen P.	LA	Α	'75
* Weinstein, Stephen A.	NY	Γ	'96
* White, R. Dudley	VA	Α	'76
* Zapor, Richard A.	CA	Ε	'84

Other

Aron, Gert	IA	Β	'59
Bernacki, Stephen E.	MA	Α	'70
Biggadike, Robert H.	AR	Α	'58
* Brule, John D.	MI	Β	'49
* Christenson, Ryan C.	UT	Β	'93
* Francois, Louise	Daughter of member		
* Harpole, George M.	CA	Ε	'74
Johnson, Roger W.	MN	Α	'79
Jones, Donlan F.	CA	Ζ	'52
Karlsson, Rolf B.	MI	Ζ	'96
Lalinsky, Mark A.	MI	Γ	'77
Lin, Sandi	MA	Β	'03
* Marks, Lawrence B.	NY	Ι	'81
Marks, Ben	Son of member		
Mazeika, Daniel F.	PA	Β	'55
* McCormick, Michael J.	Son of member		
* McHenry, S. Dale	MO	Β	'81
* Minnick, Michael V.	SC	Α	'81
Ranga, Srinivas	Non-member		
Rasbold, J. Charles	OH	Α	'83
Rausch, Mitchell T.	KS	Β	'05
Rentz, Peter E.	IN	Α	'55
Stribling, Jeffrey R.	CA	Α	'92
* Tom, Danny T.	CA	Α	'99
Valko, Andrew G.	PA	Λ	'80
Vinoski, Stephen B.	TN	Δ	'85
Voellinger, Edward J.	Non-member		
* Vogt, Jack C.	OH	Ε	'56
Wolff, Nicholas L.	NE	Α	'00
Wong, Alan	NY	Ρ	'05

* Denotes correct bonus solution

SUMMER REVIEW

The Summer column appears to have been relatively easy, with many perfect entries. The hardest regular problems were No. 3, about the area of a circle and a square, and No. 5, about radioactive iodine, with about 70 percent of the entries registering correct answers. The bonus problem was only slightly more difficult.

FALL SOLUTIONS

1 The number of goals scored by the 11 soccer players were 5, 7, 11, 13, 17, 19, 29, 31, 37, 41, and 43. There are 14 primes less than 45, the 11 listed above plus 2, 3, and 23. The sum of all 14 is 281, while the sum of the smallest 11 is 160, and the sum of the largest 11 is 271. Let A be the average score of the 11 players. Then, A must be a prime between 14.5 and 24.6, namely 17, 19, or 23. Let S be the sum of the two primes that are not used as one of the 11 scores or the average score. Then $S = 281 - 11A - A = 281 - 12A$. Substituting the three possible values for A, we get $S =$ 77, 51, or 5. So, the only solution is $S = 2 + 3$ and $A = 23$. Thus, the solution is that given above with an average score of 23.

2 The probability that a randomly chosen month will have five Sundays is 1,671/4,800 or 34.8%. The Gregorian calendar repeats itself every 400 years. Century years do not have leap years unless they are divisible by 400. A year may start with any day of the week and it may or may not be a leap year. So there are 14 different versions of a year. The cycle repeats itself every 28 years unless it includes a century year that is not a leap year, since 21(365) + 7(366) is the first cycle of years for which the total number of days is divisible by 7. Refer to a perpetual Gregorian calendar in an encyclopedia or on the Internet. Note that there are 11 versions of a year that have four five-Sunday months and only a non-leap year starting on a Sunday, and the leap years starting on a Saturday or a Sunday have five five-Sunday months. A good range to investigate is the 400 years between 1901 and 2300 because the 28-year cycle then repeats seven times until a few years before 2100. You will count a total of 43 non-leap years starting on Sunday and 28 leap years starting on Saturday or Sunday. Thus, there are $5(71) + 4(400 - 71) = 1{,}671$ five-Sunday months in a 400 year cycle.

3 There are only two ways, ignoring permutations of courts and time slots, for four couples to play a mixed-doubles tennis tournament on two adjacent courts for three consecutive time periods with no one playing with or against his or her spouse. Let the husbands be A, B, C, and D and their wives, in the same order, be E, F, G, and H. Then A plays with F, G, and H on one court while E plays with B, C, or D on the other court. This can occur six different ways. Further examination of these six ways shows that only the two schedules shown below satisfy the aforementioned restraints.

AF vs. CH	BE vs. DG
AG vs. DF	CE vs. BH
AH vs. BG	DE vs. CF
AF vs. DG	BE vs. CH
AG vs. BH	CE vs. DF
AH vs. CF	DE vs. BG

4 Alf is married to Queenie, and Bert is married to Rhona. Assume *urp* means yes and *org* means no. Then C says "B is married to Q," and P says "I am married to C," and A says "I am not married to R." P then can't be single and must be married to A or B. But, if P is married to A, A couldn't say what he did. And, if P is married to B, then C lied and is married, and A is single. But, you still don't know if Q or R is single. So, assume *urp* means no and *org* means yes. Then C says "B is not married to Q," and P says "I am not married to C," and A says "I am married to R." Then A must be single. P is either married to C, or she is single. C is either single and B is not married to Q, or C is married and B is married to Q. The "C married" case doesn't work out. But the "C is single" case yields that P is single, A is married to Q, and B is married to R.

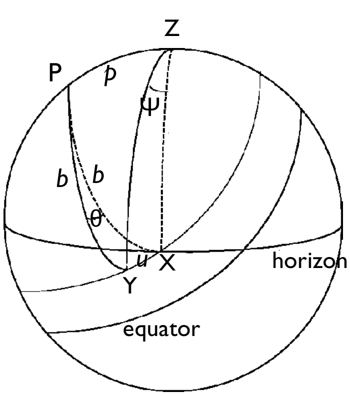
5 The solution to TWO + TWENTY = TWELVE + TEN, with the first three divisible by their namesakes, is 876 + 872480 = 872532 + 824. Note Y = 0 and T is even since TWENTY is divisible by 20. And E and O are even with similar reasoning. Then N is also even to satisfy O + Y and E + N. So W, L, and V must be odd, and two odd integers are not used. Also, E + N < 10 to satisfy WO + TY and VE + EN. Investigating O + Y and E + N yields four possibilities for values of O, Y, E, and N. Then, investigating TWO + NTY and LVE + TEN yields three possibilities. But, only 876 + 480 and 532 + 824 yields a TWELVE that is divisible by 12.

BONUS. March 21 - 1.0146 $\sin^{-1}(0.1427 \sin\phi)$ is the date of the shortest civil twilight at the latitude ϕ, with all angles in degrees. This occurs on March 15 for Leipzig (ϕ = 51° 20' 6"). This problem requires consulting spherical astronomy books. From the figure, PZX is a spherical triangle when the middle of the sun is intersected by the horizon (start of twilight), and PZY is a spherical triangle when the middle of the sun is 6.5° below the horizon (end of civil twilight). P is the earth's pole, and Z is the zenith (point directly overhead). The duration θ of twilight is XPY. Assuming the sun's declination δ (latitude with respect to the equator) is constant during the short time of twilight, one gets that PX = PY = b. Using the law of cosines for spherical triangle PXY, we get $\cos u = \cos b \cos b + \sin b \sin b \cos\theta$.

The latitude complement $b = 90° - \phi$. Since $\cos(90°-\phi) = \sin\phi$ and $\sin(90°-\phi) = \cos\phi$, $\cos u = \sin^2\phi + \cos^2\phi\cos\theta$, which leads to $\cos\theta = (\cos u - \sin^2\phi)/\cos^2\phi$. Therefore, θ is a minimum (i.e., $\cos\theta$ is a maximum) when $\cos u$ is maximum. Now consider spherical triangle XYZ. Since the middle of the sun is on the horizon at point X, ZX is 90°, and since 6.5° is the depth of the sun below the horizon at the end of civil twilight, ZY = 90°+6.5°. Therefore, by the law of cosines, $\cos u = \cos 90° \cos(90°+6.5°) + \sin 90° \sin(90°+6.5°)\cos\psi = \cos 6.5° \cos\psi$. Thus, $\cos u$ attains its maximum value when $\cos\psi$ is a maximum, i.e., when $\psi = 0$. Hence, on the day of shortest twilight, point X falls on the line ZY, and the base XY = u of the isosceles triangle PXY is 6.5°. To find the corresponding declination of the sun δ, we use the law of cosines for spherical triangles PXZ and PYZ with PZX = PZY = ω.

For triangle PXZ, $\cos b = \cos(90°-\phi) = \sin\phi = \cos p \cos 90° + \sin p \sin 90° \cos\omega = \sin p \cos\omega$. Thus, $\cos\omega$ = sinj/ sin$p$. And for triangle PYZ, $\cos b = \cos(90°-\phi) = \sin\phi = \cos p \cos(90°+6.5°) + \sin p \sin(90°+6.5°)\cos\omega = -\cos p \sin 6.5° + \sin p \cos 6.5° \cos\omega$, which gives $\cos\omega = (\sin\phi + \cos p \sin 6.5°)/(\sin p \cos 6.5°)$. Equating these two $\cos\omega$ expressions, and multiplying both sides by $\sin p \cos 6.5°$ gives $\sin\phi \cos 6.5° = \sin\phi + \cos p \sin 6.5°$ or, upon rearranging, $-\cos p \sin 6.5° = \sin\phi(1 - \cos 6.5°)$. Substituting for both $\cos p$ and $\cos 6.5°$ using the half angle formula, cos_= $1-2\sin^2(\alpha/2)$, and simplifying gives $\cos p = -\sin\phi \tan(6.5°/2)$. Because of the minus sign, p is an obtuse angle and the sun's declination, δ, is in the southern hemisphere. In this case, $p = 90° + \delta$ and $\cos p = -\sin\delta$. Thus, $\sin\delta = \sin\phi \tan(6.5°/2)$. From the declination, the sought-for day can be found (with sufficient accuracy) by means of the formula $\sin\delta = \sin(23° 27')\sin\alpha$, where λ is the angular distance of the sun from the vernal equinox (assumed to be noon, March 21). Equating the two $\sin\delta$ formulas, we get $\sin\lambda = \sin\phi \tan(6.5°/2)/\sin(23° 27')$, which gives $\lambda = \sin^{-1}[\sin\phi \tan(6.5°/2)/\sin(23° 27')]$.

Since the above-mentioned angular distance changes at an average daily rate of m = 360°/365.25 days = 59.1'/day, the sought-for information varies by $n = \lambda/m$ days from March 21. Therefore, the requested date is March 21 - λ/m or March 21 - $(365.25/360)\sin^{-1}[\sin\phi \tan(6.5°/2)/\sin(23° 27')]$ or March 21 - 1.0146 $\sin^{-1}(0.1427 \sin\phi)$. For Leipzig ($\phi$ = 51° 20' 6"), the shortest civil twilight occurs on March 15.



DOUBLE BONUS. The problem was: given the base and altitude of a triangle, find the side of the inscribed square using only a ruler and compasses. The trick to solving this problem is to realize that the infinite number of acute triangles with a given base and altitude all have the same size of inscribed square. Thus, we are free to pick the most convenient triangle, which turns out to be a right triangle. All that is needed is to construct the bisector of the right triangle and extend it until it intersects the hypotenuse. The distance from this point of intersection to either of the sides will be the length of the sides of the inscribed square. The construction shows that the base and altitude can be interchanged and not change the size of the inscribed square.

NEW WINTER PROBLEMS

1 The latest release of the Prime Players consists of three CDs, each with a different playing time. Each CD has four tracks with a total playing time of not more than one hour. The four tracks on any of these CDs each last for a different prime number of minutes. Furthermore, any combination of three different tracks on any of these CDs play for a prime number of minutes. What are the lengths of the tracks on these three CDs?

—Andrew Gibbons in *New Scientist*

2 The coordinates, in clockwise order, of the vertices of a heptagon H_1 are (0,0), (2,2), (1,4), (3,6), (3,5), (7,3), and (5,1). Form a second heptagon H_2 by joining the midpoints of adjacent sides of H_1. Next, form heptagon H_3 by joining the midpoints of the sides of H_2. It should be clear that, because these heptagons decrease in area, if this process is continued indefinitely, the heptagon will ultimately shrink to a point. What are the coordinates of this point?

—*You Are a Mathematician* by David Wells

3 Two five-person tennis teams, the Aces and the Bulls, play each other in the finals of a tournament. The ranking of the 10 players from most skillful to least skillful is A1, B1, A2, B2, A3, B3, A4, B4, A5, and B5. Any player always beats a player of lesser skill. The finals consists of five one-on-one matches, with the winner being the team that wins three or more matches. If the pairings are determined by pulling names out of a hat, what is the probability that the Bulls will win?

—**Howard G. McIlvried III,** *PA* Γ *'53*

4 Joe was playing with his seven-digit calculator. It is a calculator where each digit is represented by some combination of seven line segments, i.e., an "8" uses all seven line segments. He displayed a positive integer, and his daughter Joan looked at it upside down. She declared, "I see a number that is a perfect square, and I don't see any leading zeroes." Joe then multiplied his number by 2. Joan looked at it upside down and declared, "I see another number, and again it is a perfect square." All digits except 3, 4, and 7 are seen as digits when viewed upside down. What number did Joe originally display?

—Susan Denham in *New Scientist*

5 Three-dimensional tic-tac-toe is played on a 4 x 4 x 4 array of cells. Two players take turns by placing their symbol in any unoccupied cell. The winning player is the first one whose symbols occupy four cells in a row, either horizontally, vertically, or any of the 28 diagonals. What is the minimum number of cells that must be occupied to ensure that the other player cannot win? Submit a sketch (in pdf or jpeg format only, if your sketch is electronic) of each of the four horizontal layers that notes the positions of the occupied cells.

—**Peter E. Rentz,** *IN* Α *'55*

BONUS. The staff at a bank includes a Director, two Assistant Directors, and five Department Heads. The Director wants several different padlocks on the strongroom door with keys distributed so that he can open the door alone, either AD can open the door with the other AD or with any two DH's (but not with only one DH), or else any four DH's can open the door (but not only three DH's). All padlocks must be open to open the door. No key opens more than one lock. No lock requires more than one key to open. Keys may be duplicated to more than one person. What is the minimum number of padlocks needed, and how should the keys be distributed?

—100 Games of Logic by Pierre Berloquin

COMPUTER BONUS. A factorian is a positive integer, which is equal to the sum of the factorials of its digits. The first three factorians are 1 = 1!, 2 = 2!, and 145 = 1! + 4! +5!. What is the next factorian? Note that 0! = 1 by definition.

—*Keys to Infinity* by Clifford Pickover

Send your answers to any or all of the Winter Brain Ticklers to:
Jim Froula, Tau Beta Pi, P. O. Box 2697, Knoxville, TN 37901-2697.
If your answers are plain text only (no HTML, no attachments), email them to *BrainTicklers @tbp.org*. The cutoff date is the appearance of the Spring BENT in early April. Acknowledgments of your entry will be made in the Summer column. The details of your calculations are not required, and the Computer Bonus is not graded. We welcome any interesting new problems that may be suitable for use in this column. Jim will forward your entries to the judges:

H.G. McIlvried III, *PA* Γ *'53,*
F. J. Tydeman, *CA* Δ *'73,*
J.L. Bradshaw, *PA* Α *'82,*
and the columnist for this issue,

—**Don A. Dechman,** *TX* Α *'57.*