Brain Ticklers

RESULTS FROM WINTER

Perfect			
*Couillard, J. Gregory	IL	A	'89
*Gerken, Gary M.	CA	H	'11
*Slegel, Timothy J.	PA	A	'80
Other			
Bannister, Kenneth A.	PA	B	'82
*Bohdan, Timothy E.	IN	Γ	'85
Dechman, Don A.	TX	A	'57
*Gee, Albert	CA	A	'79
Gee, Aaron J.	CA	Ψ	'16
*Griggs Jr., James L.	OH	A	'56
*Gulian, Franklin J.	DE	A	'83
Gulian, Joseph D.	Son of member		
Handley, Vernon K.	GA	A	'86
Jenks, Timothy R.	MA	B	'13
Jordan, R. Jeffrey	OK	Γ	'00
Lalinsky, Mark A.	MI	Γ	'77
Norris, Thomas G.	OK	A	'56
Richards, John R.	NJ	B	'76
Riedesel, Jeremy M.	OH	B	'96
Roche, James R.	IN	Γ	'85
Routh, Andre G.	FL	B	'89
Schmidt, V. Hugo	WA	B	'51
*Spong, Robert N.	UT	A	'58
Strong, Michael D.	PA	A	'84
Summerfield, Steven L.	MO	Γ	'85
Voellinger, Edward J.	Non-member		
Zison, Stanley W.	CA	Θ	'87

*Denotes correct bonus solution

WINTER REVIEW

Tickler #4 (based on the topology of roads in Primeministan) turned out to be the most challenging of the problems by far. With only four correct responses, it proved even more tricky than the bonus (regarding the polka-dotted tablecloth) which had about two-thirds of the submissions correct. The easiest problem was the cryptarithm, with all submissions validating the columnist's son would endure the winter at Cornell (he did).

SPRING ANSWERS

1 Rearranged to make visualization easier, the cryptic translates to:

STRAINS	**8953078_{11}**
BRAINS	**253078_{11}**
TICKLER	**$90a6145_{11}$**

It is clear that T = S+1. Then, from the one's column, we see that R = 2S. Then, from the 10,000's column, we have C= 2R. Now, T+B=I, but T+B must have a carry; so, for example, if T=6, then we must have B>5. Also, L=2I. If you systematically go through the possible values for S and B (a spreadsheet helps), you will come to the case where S=8 and B=2, and the cryptic will look like:

$895?0?8_{11}$
$25?0?8_{11}$
$90a?1?5_{11}$

This accounts for all the digits except 3, 4, 6, and 7. These digits must form a pair that sums without producing a carry and a pair that sums while producing a carry. The only possibilities are 3+3=6 and 7+7+carry=4+11. When these values are inserted at the right locations, we arrive at the solution shown above.

2 **32** is the maximum number of knights that can be placed on a standard 8x8 chessboard so that each knight threatens exactly one other knight, as is shown in the figure:

N	N	N	N			N	N
N	N	N	N			N	N
						N	N
						N	N
N	N						
N	N						
N	N			N	N	N	N
N	N			N	N	N	N

3 The shop number is **829.** The number of strokes to paint a digit are: ZERO (5), ONE (4), TWO (4), THREE (10), FOUR (5), SIX (4), SEVEN (7), EIGHT (10), and NINE (5). Since all the digits in the shop number are different, the smallest the number of strokes could be is 4+4+4= 12, but 12 is not prime. Therefore, the smallest possible number of strokes is 4+4+5=13. Using a table of prime numbers, prepare a table consisting of three columns: column 1 is the three-digit prime numbers; column 2 is the sum of the digits in the prime (this must be a prime number, so discard rows for any primes with a sum that is not prime); column 3 is the number of strokes required for the number. If you do this, you will find that of the 30 possible shop numbers, only 829 has the sum of digits the same as the number of strokes.

4 The probability of Joe upgrading is **514/1081** or about 48%. Assume that no other cards are shown, that is, Joe has an equal chance drawing any of the 47 remaining cards. There are C(47,2) = 1081 ways of replacing the two cards Joe discarded. He can improve his hand by (a) upgrading to two pair or three of a kind in C(9,2) = 36 ways, (b) upgrade to a pair with an existing 4, 5, or 6 in C(9,1)×C(38,1) = 342 ways, (c) draw a completely new pair in 8C(4,2) + 2C(3,2) = 54 ways, (d) upgrade to a flush in C(10,2) = 45 ways, (e) upgrade to a six-high straight in C(3,1)×C(4,1) = 12 ways, a seven-high straight in C(3,1)×C(4,1) = 12 ways, an eight-high straight in C(4,1)×C(4,1) = 16 ways. A straight flush can be made in 3 ways, and these have been accounted for in both the straight and flush upgrades. So, the probability of Joe upgrading is (36+342+54+45+12+12+16–3)/1081 = 514/1081.

5 **450!** has 1001 decimal digits. It is convenient to work with the Stirling approximation for $N!$, which is $N! \approx (N/e)^N\sqrt{(2\pi N)}$. Taking logs of both sides gives $\log(N!) = N\log(N/e) + ½\log(2\pi N) = N(\log N - \log e) + ½\log N + ½\log(2\pi) = N(\log N - 0.434) + ½\log N + 0.399$. Trying a few values quickly leads to log(450!) = 1000.4. Since the number of decimal digits in a decimal integer is one more than the characteristic of its logarithm, 450! has 1001 digits.

Bonus The three triangular numbers are **10,878, 11,026,** and **11,175.** Let three consecutive triangular numbers whose product is a perfect square be $(n-1)n/2$, $n(n+1)/2$, and $(n+1)(n+2)/2$. Their product is $n^2(n+1)^2(n-1)(n+2)/8$. Since $n^2(n+1)^2/4$ is a square for all values of n, all we need do is find values for n that make $(n-1)(n+2)/2 = (n^2+n-2)/2$ a square. Let $(n^2+n-2) = 2p^2$. Multiplying by 4 gives $4n^2 + 4n - 8 = 8p^2$, which rearranges to: $(2n+1)^2 - 9 = 2(2p)^2 = 2q^2$. Rearranging again gives $r^2 - 2q^2 = 9$, where $r = 2n+1$ and $q = 2p$. This is a form of the well-known Pell equation, for which solution methods are known. Start by finding the smallest solution, which is $r = 9$ and $q = 6$, by inspection. Next, consider the equation $u^2 - 2v^2 = 1$. Again, by inspection, find the smallest solution, which is $u = 3$ and $v = 2$. Solutions to the general equation are given by successive values of $r_i - q_i\sqrt{2} = (9 - 6\sqrt{2})(3 - 2\sqrt{2})^x$ as x goes from 0 to ∞. When x=0, r=9 and n= $(9-1)/2$=4, the three smallest triangular numbers that fit the conditions of the problem are 6, 10, and 15, and the square equal to their product is $6(10)(15)=900=30^2$. When x=1, r=51 and n=25, the three triangular numbers are 300, 325, and 351, and their product $300(325)(351) = 34{,}222{,}500 = 5{,}850^2$. When x=2, r=297 and n=148, the three triangular numbers whose product is a square are 10,878, 11,026, and 11,175; their product is $10878(11026)(11175) = 1{,}340{,}338{,}752{,}900 = 1{,}157{,}730^2$.

Computer Bonus **357,686,312,646,216,567,629,137** is the largest decimal prime integer with the property that as the leading digit is successively removed, the resulting integers are still all primes.

NEW SUMMER PROBLEMS

1 After a church service, a priest and a cantor had a conversation. The priest asked, "Do you know the ages of the three visitors we had today?" The cantor said, "I don't know." The priest said, "What if I told you that the product of their ages is 2652?" The cantor did some calculations and then replied, "I still don't know." The priest then leans in and whispers another hint into the cantor's ear. The cantor responded, "I still don't have enough information." The priest said, "By the way, thank you for that book of Brain Ticklers you gave me for my birthday. I see by the card you sent me that you know how old I am. I wonder if those three visitors will still enjoy challenging their minds when they reach my age." The cantor then said, "Oh! Now I know their ages and anyone listening to us should still be able to determine your age." Assuming that all ages are integral, how old is the priest?

—Grant L. Bilker, son of **Scott L. Bilker,** *NJ Γ '91*

2 You are playing a game where you toss a circular disk randomly on a floor made up of square tiles with sides of unit length. You win if your disk lands so as to cover parts of at least two tiles, as long as it doesn't cover a corner of a tile. You get to pick the diameter of the disk. What is the maximum probability of winning and what diameter disk achieves this?

—Brian Barwell and **Richard Hess,** *CA B '62*, in *Mental Gymnastics*

3 How many ways are there of walking up a flight of 13 stairs if you take either 1 or 3 stairs with each step?

—*Eustice Solves a Problem*

4 Jan has 27 wooden 1-inch cubes whose faces have been painted with red, white, and blue paint. Using these, she can assemble a 3-inch cube that is red on all sides. She can rearrange them to form a 3-inch cube that is white on all sides. She can also arrange them to form a 3-inch cube that is blue on all sides. How many of the 1-inch cubes have 2 red sides, 2 white sides, and 2 blue sides?

—Tom Stluka, brother of **Frederick Stluka,** *PA Δ '82*

5 In the poker game of showdown, each player is dealt one card face up from a well shuffled standard deck of 52 cards and the highest card wins. If there are five players, exactly which card would give a player just over a 50% chance of winning the pot? Assume the hierarchy of suits in descending order is spades, hearts, diamonds and clubs.

—Don Dechman, *TX A '57*

Bonus Three logicians, A, B, and C, each wears a hat with a positive integer on it. Each logician sees the numbers on the other hats, but not their own. Each knows the numbers are positive integers and that one is the sum of the other two. They take turns in order, starting with A, to identify their number, or pass if they can not. A, B, and C all pass on their turn in each of the first two rounds. In the third round, A and B pass, but C correctly asserts her number is 89. What are the other two numbers? Provide a quick outline of C's logic used to determine her number was 89.

—Donald Aucamp, Puzzle Corner in *Technology Review*

Computer Bonus Consider the sequence of rational fractions, 3/1, 13/4, 16/5, 19/6, 22/7, 179/57, 201/64, 223/71, 245/78, 267/85, 289/92, 311/99, 333/106, 355/113, 52163/16604,... P_i/Q_i..., of successively closer approximations to π. In this sequence, each entry has a smaller error than any other entry with a smaller value of Q_i. A ratio P_i/Q_i is a member of the sequence if, and only if, it meets this criterion. Thus, this sequence includes all the convergents resulting from expressing π as a continued fraction, but many other values as well. Find the smallest P_i such that its units digit is 0, the units digit of P_{i+1} is 1, the units digit of P_{i+2} is 2, etc. up to the units digit of P_{i+7} is 7. Present your answer as the first value of this series.

—Timothy J. Slegel, *PA A '80*

Postal mail your answers to any or all of the Brain Ticklers to **Tau**

(Continued on page 44)