Brain Ticklers

RESULTS FROM WINTER

Perfect		
*Bohdan, Timothy E.	IN Γ	'85
Brule, Mark	Son of member	
*Couillard, J. Gregory	IL A	'89
Garcia, Agustin M.	NY O	'80
Jones, John F.	WI A	'59
Jones, Jeffery C.	Son of member	
Kenny, Thomas A.	NY Θ	'89
Kimsey, David B.	AL A	'71
*Norris, Thomas G.	OK A	'56
Prince, Lawrence R.	CT B	'91
*Schmidt, V. Hugo	WA B	'51
*Slegel, Timothy J.	PA A	'80
Slegel, Craig M.	Son of member	
*Spong, Robert N.	UT A	'58
*Strong, Michael D.	PA A	'84
Other		
Alexander, Jay A.	IL Γ	'86
Aron, Gert	IA B	'58
Beaudet, Paul R.	Father of member	
*Celani, Paul E.	MD Γ	'97
Colbourne, Richard J.	PA E	'78
*Crawford, Martin	TN A	'54
Handley, Vernon K.	GA A	'86
*Johnson, Mark C.	IL A	'00
Johnson, Roger W.	MN A	'79
Jones, Donlan F.	CA Z	'52
Lalinsky, Mark A.	MI Γ	'77
Maritch, Mark J.	PA E	'03
Marrone, James I.	IN A	'61
Parks, Christopher J.	NY Γ	'82
Pendleton III, Winston	MI Γ	'62
Quan, Richard	CA X	'01
Rentz, Peter E.	IN A	'55
Richards, John R.	NJ B	'76
Riedesel, Jeremy M.	OH B	'96
Roggli, Victor L.	TX Γ	'73
Schweitzer, Robert W.	NY Z	'52
Sigillito, Vincent G.	MD B	'58
*Skowronski, Victor J.	NJ A	'71
*Smith, Ronald E.	PA A	'86
Stadlin, Walter O.	NJ Γ	'52
Stribling, Jeffrey R.	CA A	'92
Summerfield, Steven L.	MO Γ	'85
Sutor, David C.	Son of member	
Svetlik, J. Frank	MI A	'67
Vegeais, James A.	IL A	'86
*Verkuilen, William W.	WI B	'92
Voellinger, Edward J.	Non-member	
*Widmer, Mark T.	OH A	'84

* Denotes correct bonus solution

WINTER REVIEW

The most difficult regular problems were No.1, about green cubes touching a red cube, and No. 3, about the number of times the calendar date uses the digits 1 to 9 each once, with about 2/3 of the answers being correct. Only about half the answers to the Bonus, about BINGO odds, were correct.

SPRING SOLUTIONS

Readers' entries for the Spring Ticklers will be acknowledged in the Fall *BENT*. Meanwhile, here are the answers:

1 TRY×THIS = TICKLER translates to **986x9203 = 9074158**. This is a fairly difficult cryptic. One approach, which uses a spreadsheet, is as follows. Start by assuming T = 9, then look at possible values for Y and S and generate a table giving Y*S mod(10) which is equal to R. Then, generate a second table which is TRY for possible values of Y (TRY = 900 + 10×R + Y). Next, pick H and I and generate another table with values for THIS (THIS = 9000 + 100×H + 10×I + S). Finally, create a table which calculates TRY×THIS=TICKLER. Examine the entries in the table to find values of TICKLER which start with 9, have a second digit equal to I and have all 7 digits different. It isn't too difficult to find the answer.

2 **$7^{2048} - 1$ is divisible by 4^7.** Start by factoring $(7^{2048}-1) = (7-1)(7+1)(7^2+1)(7^4+1)\ldots(7^{1024}+1)$. It is obvious that 7^y+1 is divisible by 2 for all positive integral y. Now, $7^2 = 49 \equiv 1 \pmod 4$. Squaring gives $7^4 \equiv 1 \pmod 4$; squaring again gives $7^8 \equiv 1 \pmod 4$, or in general, $7^m \equiv 1 \pmod 4$, where $m=2x$, with x a positive integer. Adding 1 gives $7^m+1 \equiv 2 \pmod 4$; thus 7^m+1 is divisible by 2 but not by 4. The factorization of $(7^{2048}-1)$ has 10 such factors plus the factors (7–1) and (7+1); (7+1) is divisible by 2^3, while (7-1) is divisible by 2. Thus, $(7^{2048}-1)$ is divisible by $2^{10+3+1} = 2^{14} = 4^7$, so $n = 7$. An alternative approach to the solution is to rewrite $7^{2048}-1$ as $(8-1)^{2048}-1$, and after binomial expansion, consider the divisibility of each addend.

3 The colors and corresponding sizes of the wooden cubes are: **a=4, b=9, c=11, d=1, e=3, f=2, g=12, h=10, i=8, j=7, k=6, l=5**. Let uppercase letters A through L represent the respective volumes of the twelve wooden cubes. Each side is an integral number of cm, so the possible values for the volumes are 1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, 1331, and 1728. The twins discovered balances represented by the following four equations:

(1) D + G = B + H
(2) K = A + E + L
(3) E + G + J + K = A + B + D + F + H + I
(4) B = D + I + K

Considering equation (1), there are only two pairs of volumes that have the same sum: 1 + 1728 = 729 + 1000. From (4), we know that B > D, so the quadruple (B, D, G, H) must be one of (1728, 1000, 729, 1), (1728, 729, 1000, 1), (1000, 1, 1728, 729) or (729, 1, 1728, 1000).

Rearranging (4), we see that B - D = I + K, so I + K must be one of 728 or 999. No two volumes sum to 999, and only 216 + 512 = 728, so I, K must be 216 and 512 in some order. Therefore, we know (B, D, G, H) is either (1728, 1000, 729, 1) or (729, 1, 1728, 1000).

Now, examining equation (2), we must choose three of 8, 27, 64, 125, and 343 to sum to either 216 or 512. K cannot be 512, because no combination of three of the remaining five cubes sum properly. But since 216 = 27 + 64 + 125, K = 216, I = 512, and A, E, L are each one of 27, 64, and 125.

Subtracting (1) from (3) and replacing K and I with their known values, we get (5) E + J = A + 2D + F + 296. We know that: A and E come from the set of 27, 64 and 125; F and J come from the remaining values 8, 343 and 1331; and D is either 1 or 1000. There's no E and J large enough for D = 1000 in (5), so D = 1 (which implies B = 729, H = 1000, G = 1728) and (5) reduces to E + J = A + F + 298. We resort to trial and error to determine E = 27, J = 343, A = 64, F = 8. Then, by default L = 125, and C = 1331.

4 Jill made a **5x5x5 cube and painted one face;** Jackie built a **6x6x6 cube and painted four faces.** The number of small unit cubes making up a large cube with an edge length of n is n^3, and the number of internal small cubes = $(n-2)^3$. Now, $(n-2)^3 <= 100$, so $n < 7$, and $n^3 > 100$ so $n > 4$, therefore $n = 5$ for Jill and $n = 6$ for Jackie. For a given n, the number of external blocks equals $n^3 - (n-2)^3$, and the number of unpainted external blocks is $100 - (n-2)^3$, so the number of painted blocks is $n^3 - 100$. For $n = 5$, the number of painted blocks is $5^3 - 100 = 25$, which means that Jill painted exactly one face of her large cube. A similar calculation for Jackie's 6-cm cube shows that $6^3 - 100 = 116$ of her small cubes were painted, that is, $100 - 4^3 = 36$ external cubes were unpainted. Allowing for small cubes along the edges, 2 adjacent faces of the large cube were left unpainted, so exactly 4 faces of the large cube were painted.

5 Three passes through the copier at magnifications of **125%, 125%, and 128% will enlarge the linear dimensions of a picture by a factor of exactly 2.** We are looking for a sequence of five (or fewer) factors f_i whose product is exactly 2. Each f_i is an integral percentage greater than 1, but no more than 1.55. Each f_i can be represented as a rational fraction $F_i/100$, where F_i is an integer between 101 and 155 inclusive. Let p equal the product of these fractions. The denominator of p is $100^n = 2^{2n}5^{2n}$, where n is the number of passes through the copier. Since the denominator consists of an equal number of 2's and 5's, and since p is exactly 2, the prime factorization of the product of the F_i's must be $2^{2n+1}5^{2n}$, that is, one more 2 than 5 and no other factor. The only integers between 101 and 155 whose only prime factors are 2's and/or 5's are $125 = 5^3$ and $128 = 2^7$. For $n = 3$, one factor of 128 and two factors of 125 give $2^7 5^6$, one more 2 than 5, so the sequence of 125%, 125%, 128% gives an enlargement by a factor of exactly 2.

Bonus. The tenth smallest integer which is simultaneously the sum of two and three consecutive squares is **302,991,312,620,897,818,205,** and the ratio of the $(n+1)$st such number to the nth such number is **$49+20\sqrt{6}$** $= 97.98979486$. The equation for numbers of the type considered here can be written as $N = n^2 + (n+1)^2 = (m-1)^2 + m^2 + (m+1)^2$. Combining terms and multiplying by 2 yields $4n^2 + 4n + 2 = p^2+1 = 6m^2 + 4$, or $p^2+1 = 6m^2 + 4$. Rearranging gives the Pell equation $p^2 - 6m^2 = 3$, where $p = 2n+1$. A solution to this equation gives m, and m gives $N = 3m^2+2$. Number theory shows that, given a solution to this Pell equation, an infinite number of other solutions can be generated. We need both a solution to $p_0^2 - 6m_0^2 = 1$ and a solution to $r^2 - 6s^2 = 3$. Then $(p_0+m_0\sqrt{6})^n(r+s\sqrt{6}) = (p_n+m_n\sqrt{6})$ will generate as many solutions (p_n, m_n) as we desire. By inspection, $5^2- 6(2^2) = 1$, and $3^2 - 6(1^2) = 3$, so $(p_0, m_0) = (5, 2)$, and $(r, s) = (3, 1)$. Then, $p_1 = 5(3)+6(2)(1) = 27$ and $m_1 = 2(3)+5(1) = 11$, so $N_1 = 3m_1^2+2 = 3(11^2)+2 = 365$. Next, we get $p_2 = 5(27)+6(2)(11) = 267$, $m_2 = 2(27)+5(11) = 109$, and $N_2 = 3(1092)+2 = 35,645$. Continuing to n=10, we get the 21-digit number given above. To determine the ratio of N_{n+1}/N_n, start by determining the ratio for p_n and m_n; this ratio is: $R = (p_0+m_0\sqrt{6})^{n+1}(r+s\sqrt{6})/(p_0+m_0\sqrt{6})^n (r+s\sqrt{6}) = (p_0+m_0\sqrt{6}) = (5+2\sqrt{6})$; but this is the ratio for m, and since N is a function of m^2, we need the square of this value, or $R = (5+2\sqrt{6})^2 = 49+20\sqrt{6}$.

Computer Bonus The two smallest bases such that ba exactly divides $abcde$ are **1182** and **14,194.** Converting $abcde_n$ and ba_n into decimal notation gives $10n^4+11n^3+12n^2+13n+14$ and $11n+10$, so what we want is $(10n^4+11n^3+12n^2+13n+14)/(11n+10)$ to be an integer. When n=1182, we have 19,537,736,900,876/13,012= 1,501,516,823 and for n=14,194, then 405,931,600,977,692,352/156,144= 2,599,725,900,308. It is quite surprising that we would find a solution at such large bases.

Double Bonus The # operation we had in mind is to **add the two numbers on either side of # and subtract from 10.** Thus, 4 # 4 = 2. Finding an alternate solution seems to be limited only by the ingenuity of the solver.

NEW SUMMER PROBLEMS

1 The planet Topsis is unusual in that each "hemisphere" is a right-circular cone of height 3000 km and base radius 4000 km. The two cones are joined base to base, so that the equator is the base circle and the vertical distance between the poles is 6000 km. Topsy and Turvy are two cities located on the planet's equator 180 degrees apart. To the nearest km, what is the shortest distance between the two cities, assuming that one must travel along the surface of the planet?

—*Superior Mathematical Puzzles* by Howard P. Dinesman

2 A fair coin is flipped ten thousand times. What is the expected value of the absolute difference between the number of heads and the number of tails? Express your answer to four significant figures.

—Kevin M.T. Stewart, *NJ Δ '77*

3 The digits 1, 2, and 3 can be combined in 27 different ways to form a three digit number, e.g., 111, 112, 123, 232, 321, …, 333. Arrange nine 1's, nine 2's and nine 3's around a circle so that each of the 27 combinations of these three digits appear exactly once when reading the arrangement in a clockwise direction. For example, the sequence 1112321 would give the first five values above. Express your answer as a 27 digit number, starting with 1112 and proceeding around the circle.

—*Problem Solving Through Recreational Mathematics* by Bonnie Averbach and Orin Chein

4 Five soccer playing countries were entered in 2016 European Consolation Cup and played the usual one game against each of the others. Each nation played two games at home, two games away, and there were no draws. Exactly three of the games were won by the away side. No two countries won the same number of games. Azerbaijan won two. Belarus was away at Azerbaijan and at Cyprus. Cyprus played away at Azerbaijan. Denmark was home against Estonia. How many games did each nation win?

—A Tantalizer by Martin Hollis in *New Scientist*

5 Starting with 1, write the positive integers in continuous order without separating them as if you were writing a single number: 1234567891011121314... The first and tenth digits in this number each are clearly 1. In addition to those first two, what are the hundredth, thousandth, …, through trillionth digits in this number? That is, what digits occupy places 10^n, for n=0 through 12? Express your answer as a sequence of 13 digits.

—*Mathematical Recreations* by Maurice Kraitchik

Bonus Alf, Bert, Charlie, and Doug were the first to volunteer as the crew for the British spacecraft. At the time with which our story is concerned, they were moving through space at a speed, and in a direction, such that the hands of their special chronometer, which allowed for local variations in time, had, since liftoff, been moving exactly twice as fast as those of the Greenwich Mean Time (GMT) clock. In the following exchange, all references to the passing of time as recorded by the GMT clock are in italics (*time*, *minutes*, *hours*, etc.), and all references to time recorded by the ship's chronometer are in a regular font (time, minutes, hours, etc.). The four space travelers speak, with their statements starting at *5 minute* intervals:

Doug: Oh, Alf! Oh, Bert! That all should come so clear! In less than half an hour, tomorrow's here, and our today is one with yesteryear.

Alf: Yes, your today perhaps, but not so mine! You watch the clock; myself, I watch the *time*. *10 minutes* since I heard the *hour* sound; and when another *23 minutes* have passed, then 2½ hours will have elapsed, since that moment we boarded, 34 minutes before we left the ground.

Bert: *Tomorrow* and tomorrow and *tomorrow*! When will they both arrive? In more than *2 hours*. Alas! Before the *time* is what the time is now, only a *half an hour* must pass. Oh, Alf and Charlie, ere then I do pray, repent of the lies you've told this day. (In case I fail to make my meaning plain, you're liars both. Try not to lie again!)

Charlie: I'm not a very clever chap. I cannot tell the time. I cannot add, nor yet subtract, much less compute a prime. I never learnt to cut a rug, some say that I'm uncouth. But unlike those liars Bert and Doug, Alf and I do tell the truth.

Everything a speaker says is either all true or else everything he says is false. Prior to launch, the ship's chronometer is synchronized with GMT, and each speaker knows the veracity of every other speaker. What was the *time* (GMT) when they lifted off?

—*Brain Puzzlers Delight* by E.R. Emmet

Computer Bonus In 2015, after a long career teaching mathematics, Dr. E. Trig retired. It was his habit each year to ask his students to find the smallest number which, when multiplied by the year, would result in an integer all of whose digits were either 0's or 1's. For example, 5,568,978 times 1995 equals the 11-digit number 11,110,111,110. In 1945, the first year he asked, most of his students were able to find a suitable four digit number. Of the years for which Dr. Trig posed his question, which one had the longest required number and what was that number?

—An Enigma by Susan Denham in *New Scientist*

Send your answers to any or all of the Summer Brain Ticklers to Curt Gomulinski, Tau Beta Pi, P. O. Box 2697, Knoxville, TN 37901-2697 or email to *BrainTicklers@tbp.org*, plain text preferred. The cutoff date for entries to the Summer column is the appearance of the Fall *BENT* in late September (the digital distribution is a few days earlier). The method of solution is not necessary. The judges welcome any interesting problems that might be suitable for the column. The Computer Bonus is not graded. Curt will forward your entries to the judges who are **H.G. McIlvried III,** *PA Γ '53;* **F.J. Tydeman,** *CA Δ '73;* **D.A. Dechman,** *TX A '57;* and the columnist for this issue, **J.C. Rasbold,** *OH A '83.*