Brain Ticklers

RESULTS FROM WINTER

Perfect

*Bohdan, Timothy E.	IN	Γ	'85
*De Vincentis, Joseph W.	TX	Γ	'93
*Gerken, Gary M.	CA	Η	'11
*Harvey, Arthur J.	OH	Α	'83
Janssen, James R.	CA	Γ	'82
Kimsey, David B.	AL	Α	'71
Riedesel, Jeremy M.	OH	Β	'96
*Slegel, Timothy J.	PA	Α	'80
Slegel, Craig M.	Member's son		
Stein, Gary M.	FL	Δ	'04
*Strong, Michael D.	PA	Α	'84

Other

Aron, Gert	IA	Β	'58
Beaudet, Paul R.	Member's father		
Brule, John D.	MI	Β	'49
*Couillard, J. Gregory	IL	Α	'89
Crawford, Martin	TN	Α	'54
Epping, Cole	Son of member		
*Gibbs, Kenneth P.	MO	Γ	'76
Hasek, William R.	PA	Γ	'49
*Hutchison, Dylan	NJ	Α	'15
Jones, Donlan F.	CA	Ζ	'52
Kimmel, Peter G.	Member's spouse		
Lalinsky, Mark A.	MI	Γ	'77
Marrone, James I.	IN	Α	'61
*Norris, Thomas G.	OK	Α	'56
Parks, Christopher J.	NY	Γ	'82
*Peters, Paul E.	KS	Α	'57
Preble, Harry L.	MA	Δ	'61
Rentz, Peter E.	IN	Α	'55
*Richards, John R.	NJ	Β	'76
Schmidt, V. Hugo	WA	Β	'51
Schweitzer, Robert W.	NY	Ζ	'52
Sedlak, Matthew	NY	Γ	'78
Sigillito, Vincent G.	MD	Β	'58
Spong, Robert N.	UT	Α	'58
*Stephenson, G. Brian	MA	Β	'78
Stetson II, Scott B.	CA	Τ	'12
*Stribling, Jeffrey R.	CA	Α	'92
Summerfield, Steven L.	MO	Γ	'85
Sutor, David C.	Member's son		
*Thiele, Karl E.	NY	Γ	'82
Van Sickel Jr, James R.	OH	Η	'77
Vinoski, Stephen B.	TN	Δ	'85
Voellinger, Edward J.	Non-member		

*Denotes correct bonus solution

WINTER REVIEW

Problem #1 about finding six unique six-digit numbers that contain only two different integers was the most tricky and caused several entries from being "perfect".

SPRING SOLUTIONS

Readers' entries for the Spring Ticklers will be acknowledged in the Fall column. Meanwhile, here are the answers:

1 The probability of drawing four doubles in a hand of 11 dominos with a double 12 set is 0.0399. The number of dominos in a double N set is $\sum_{i=1}^{N+1} i = (N+1)(N+2)/2 = 91$ for double 12. The number of doubles is $N+1 = 13$. The number of ways to get four doubles is C(13, 4), where C(i, j) is the number of combinations of i objects taken j at a time. The number of ways of selecting the other 7 dominos is C(78, 7), and the number of different 11 domino hands is C(91, 11). Therefore, the probability of getting four doubles is P = C(13, 4)C(78, 7)/C(91, 11) = 715 x (2,641,902,120)/ 47,325,339,895,743 = 0.03991434652.

2 LETTERS + ALPHABET = SCRABBLE translates to **7088062 + 17531908 = 24619970**. From E+B=B, we deduce E=0, so R+E=L means L=R+1, and E+P=R means R=P+1, so P, R, L have three consecutive values. Also, S+T=E means S+T=10, and since S=A+1, T+A=9, so B=9. Now, L cannot be less than 5, since there must be a carry for S>A. However, L≠5, as L=5 would make C=0, but E=0; and L≠6 as L=6 means C=2. Then, A≠1 as A=1 would make S=2, but C=2; and A≠3, for A=3 means S=4, but P=4; and A≠4 as P=4; and A≠5 as R=5; and A≠7 for A=7 means S=8 and T=2, but C=2; and A≠8 as A=8 means S=9, but B=9. Thus, L≠6. Also, L≠8, as L=8 makes C=6, but P=6, so L≠8. Thus, L=7, R=6, P=5, C=4, A=1 (A≠2, for if A=2, S=3 and T=7, but L=7), S=2, T=8, and H=3, which produces the unique solution shown above.

3 The speed of the golf ball as it passes the center of the moon is **6,048 km/h**. Start with the fundamental equation of motion, $F = ma$. In our case, F is gravitational force equal to GMm/R^2, where G is the gravitational constant, M is the mass of the moon, m is the mass of an object on the moon's surface, and R is the radius of the moon. Let ρ equal the density of the moon, then $M = (4/3)\pi\rho R^3$, but $g_m = (4/3)\pi\rho GR^3/R^2 = (4/3)\pi\rho GR$. Inside a sphere, only the mass within a radius equal to the position of an object and the center contributes to gravity, so inside the moon, $a = -(4/3)\pi\rho Gr = -g_m r/R$, where r is the distance from the golf ball to the center of the moon. Now, $a = d^2r/dt^2 = d(dr/dt)/dt = dv/dt = (dv/dr)(dr/dt) = v(dv/dr) = -g_m r/R$. This gives $v dv = -(g_m/R) r dr$ or $\int_0^v v dv = -(g_m/R)\int_R^0 r dr$, which yields $[v2/2]_0^v = -(g_m/R)[r^2/2]_R^r$ or $v^2 = (g_m/R)(R^2-r^2)$. At the center of the moon, $r = 0$, so $v_c = \sqrt{[(g_m/R)(R^2)]} = \sqrt{(g_m R)} = [(1.6249)(1,737,000)]^{0.5}$ = 1,680 m/s = 6,048 km/h.

4 Prof. Adams's combination is **49236**. Possible squares are 25, 36, 49, and 64 (16 and 81 are excluded as the smallest digit is in the middle). The possible combinations are shown in the table. Since there are five digits, there are 5! = 120 permutations of the keypad combination. When they are added, each column (ones, tens, hundreds, thousands, and ten thousands) will contain each digit 120/5 = 24 times. Thus, if the sum of the digits is S, each column will sum to 24S, and the total sum will be (24+240+2,400+24,000+240,000)S = (24)(11,111)S = 266,664S. The table also shows the digit sums and the corresponding permutation sums. Only, 49236 gives a palindrome all of whose digits are divisible by 3.

Combination	Sum of Digits	Sum of Permutations
36025	16	4,266,624
36125	17	4,533,288
49025	20	5,333,280
49125	21	5,599,944
49036	22	5,866,608
49136	23	6,133,272
49236	24	6,399,936
64025	17	4,533,288
64125	18	4,799,952

5 The problem concerning connecting six points with nine line segments and avoiding creating a triangle can be solved by trial and error or by picking any three of the points and designating them as "from" points, and designating the other three points as "to" points. Now, construct line segments from each of the "from" points to each of the "to" points. There are C(6, 3) = 20 possibilities, but switching the "to" and "from" points doesn't change the result, so there are really 20/2 = 10 solutions, and all are valid; each must have three line segments meeting at each point. We owe thanks to Fred Tydeman for providing this insightful analysis of this Tickler. We gave credit for any correct answer.

Bonus The expected value is $[\sqrt{2} + \ln(1 + \sqrt{2})]/3$. Consider a square with its lower left hand corner at the origin of a coordinate system, and draw a diagonal through the origin. Because of symmetry, we need consider only half the square, but it is necessary to introduce a factor of 2 to account for this. It is easier to use polar coordinates, where the element of area is $r\,dr\,d\theta$, and r is the distance to the origin. We need to evaluate the double integral $E = 2\int_0^{\pi/4}\int_0^{\sec\theta} r^2\,dr\,d\theta = 2\int_0^{\pi/4}[r^3/3]_0^{\sec\theta} = (2/3)\int_0^{\pi/4}\sec^3\theta\, d\theta = (1/3)[\tan\theta\sec\theta + \ln(\tan\theta + \sec\theta)]_0^{\pi/4} = [\sqrt{2} + \ln(1 + \sqrt{2})]/3 = 0.7652$. If you have forgotten how to evaluate the integral of $\sec^3\theta$, you can look it up in a table of integrals or on the internet.

Double Bonus Let $s_{11}, s_{12}, s_{13}, s_{21}, s_{22}, s_{23}, s_{31}, s_{32}, s_{33}$ be the 9 cells of a 3x3 magic square. You were asked to find such a square in which $s_{11}s_{12}s_{13} + s_{21}s_{22}s_{23} + s_{31}s_{32}s_{33} = s_{11}s_{21}s_{31} + s_{12}s_{22}s_{32} + s_{13}s_{23}s_{33} = s_{11}s_{22}s_{33} + s_{13}s_{22}s_{31}$. It helps to start with a generalized 3x3 magic square.

$a + b$	$a - b - c$	$a + c$
$a - b + c$	a	$a + b - c$
$a - c$	$a + b + c$	$a - b$

For any values of a, b, and c, this square will be magic with a magic sum of $3a$. Performing the product/sum calculation using this square shows that the sums for the rows and columns are always equal and have the value $3a(a^2 - b^2 - c^2)$, while for the diagonals it is $a(2a^2 - b^2 - c^2)$. Setting these equal and simplifying gives $a^2 - 2b^2 - 2c^2 = 0$. Multiplying by 2 and rearranging gives $p^2 - a^2 = a^2 - q^2$, where $p = 2b$ and $q = 2c$, which shows that p^2, a^2, and q^2 are three squares in arithmetic progression, and any such squares lead to a magic square with the desired property. These squares satisfy the relationship $p^2 + q^2 = 2a^2$, or $(p/a)^2 + (q/a)^2 = 2$, so $(p/a, q/a)$ must be a rational point on the circle $u^2 + v^2 = 2$, and by proper choice of k, $(ku)^2$, k^2, $(kv)^2$ is an arithmetic progression of integral squares. Let (u_i, v_i) be a point on the circle, and construct line $v = mu + n$ through (1, 1) and (u_i, v_i). Since the point (1, 1) satisfies the line, we get $n = 1 - m$. To find u_i and v_i in terms of m, substitute $v_i = mu_i + 1 - m$ into the circle equation to give: $(m^2 + 1)u_i^2 + 2m(1 - m)u_i + (m^2 - 2m - 1) = 0$. Solving for u_i gives $u_i = (m^2 - 2m - 1)/(m^2 + 1)$, which, when substituted into the line equation gives $v_i = (-m^2 - 2m + 1)/(m^2 + 1)$, so (u_i, v_i) are rational points on the circle. The three squares in arithmetic progression are: $(m^2 - 2m - 1)^2$, $(m^2 + 1)^2$, $(-m^2 - 2m + 1)^2$, or letting $m = r/s$ and multiplying by s^2 gives: $(r^2 - 2rs - s^2)^2$, $(r^2 + s^2)^2$, and $(-r^2 - 2rs + s^2)^2$ as the three squares. The values of a, b, and c that satisfy the magic square are: $a = 2(r^2 + s^2)$, $b = (r^2 - 2rs - s^2)$, and $c = (s^2 - 2rs - r^2)$. All the fundamentally different magic squares that satisfy the specified conditions can be generated by letting r/s equal successively: ½, 1/3, 2/3, ¼, ¾, 1/5, 2/5, 3/5, 4/5, … where r and s are relatively prime, and calculating the cell values in the magic square (see second figure). From symmetry, it is clear that $m = s/r$ will give the same result as $m = r/s$. Other solutions can be obtained by multiplying a fundamental solution by any positive integer.

$(r - s)^2 + 2r^2$	$2(r + s)^2$	$(r - s)^2 + 2s^2$
$4s^2$	$2(r2 + s^2)$	$4r^2$
$(r + s)^2 + 2r^2$	$2(r - s)^2$	$(r + s)^2 + 2s^2$

The same result can be derived using Pythagorean triples. If the two legs of a right triangle are x and y, and the hypotenuse is z, then $a = 2z$, $b = x + y$, and $c = x - y$ produces a solution. Any book on number theory will give the following formula for deriving a pythagorean triple: $x = 2sr$, $y = s^2 - r^2$, and $z = s^2 + r^2$, where s and r are relatively prime positive integers of opposite parity with $s>r$, so a $= 2(s^2 + r^2)$, b $= 2sr + s^2 - r^2$, and c $= 2sr - s^2 + r^2$. Substituting these values into the first figure gives a version of the second figure. This interesting problem, which involves magic squares, squares in arithmetic progression, rational points on a circle, and Pythagorean triples, shows the deep interconnections of mathematics.

NEW SUMMER PROBLEMS

1 Our local baseball park is rather oddly shaped. It is exactly 260 feet from first and third bases to the foul poles and from second base to the fence in dead center field. The outfield fence itself is in the form of an arc of a circle. In order to improve the production of home runs, it has been decided to reconstruct this fence so that the fair ball portion of the field will be in the shape of a quadrant of a circle with a radius of 350 feet (distance from home plate to the outfield fence). Thus, the distance from home plate to the fence along the foul lines will be the same as before, but the distance from home to the center field fence will be shorter. How much area will be removed from the fair ball portion of the park? Express your answer to the nearest square foot.

—Larry M. Lesser, son of member

2 A jeweler dies and bequeathes his hoard of 45 jewels (9 diamonds, 9 rubies, 9 sapphires, 9 opals, and 9 zircons) to his five children. The diamonds are worth $1,000 each, the rubies $750, the sapphires $500, the opals $250, and the zircons are worthless. His will specifies that the jewels be divided so that each child gets the same total value and the same total number of jewels. Furthermore, each child gets at least one of each kind of gem, but that no two children get identical combinations of gems. Finally, each kind of gem is to be distributed as evenly as possible. That is, the total number of three-of-a-kinds among the five children is to be a minimum. How should the jewels be distributed?

—*Mathematical Recreations*, by Maurice Kraitchik

3 On the game show *What's in a Name?*, N contestants write their names on slips of paper and drop them into a bowl. The slips are mixed, and each person draws a slip. If a contestant gets his own name, he drops out. The remaining contestants put their slips back in the bowl, and the process is repeated. The prize money is split evenly among those going out on the last round (which leaves no players). What is the expected number of rounds for everyone to have dropped out? For large values of N, what is the probability that the total prize is split between exactly two contestants?

—*Mathematics Magazine*

4 Four logicians went on a camping trip. Among their supplies were 11 donuts. At their evening campfire, each logician ate one donut. During the night, one or more of the logicians, unbeknownst to the others, raided the donuts, so that in the morning the box was empty. The following exchange then took place. Ann asked, "Beth, did you eat more donuts than I did?" Beth replied, "I don't know. Carol, did you eat more donuts than I did?" Carol answered, "I don't know." Diane then remarked, "Now I know how many donuts each of us ate." If no logician asked a question to which she knew the answer, how many donuts did each logician eat? Assume the logicians are honest, at least when it comes to answering questions truthfully.

—*Hard-to-Solve Brainteasers*, by Jaime and Lea Poniachik

5 Consider six one-ohm resistors that are connected to form a tetrahedron. What is the resistance between any two vertices?

—Allan Gottlieb's Puzzle Corner in *Technology Review*

Bonus Two Tau Bates play a game. They start with a sheet of paper on which are written the integers from 1 to 100 (inclusive). They take turns crossing off numbers. Once a number is crossed off, it is dead and cannot be used again. Other than the opening move, the chosen number must be either an exact divisor of the previous choice or an exact multiple of it. For example if one player chose 45, the other player could choose 1, 3, 5, 9, 15, or 90 (provided they were still available). The opening move must be an even number. The first player who cannot follow the rules loses. Can the player crossing off the first number force a win? If so, what opening move should be made?

—*Professor Stewart's Hoard of Mathematical Treasures*, by Ian Stewart

Computer Bonus Arrange the integers from one to N in alphabetical order. To avoid ambiguity, use no hyphens or and's. Consider spaces to alphabetically precede all letters. For example, 837,301 would be spelled eight hundred thirty seven thousand three hundred one; also, 1,807 is one thousand eight hundred seven, not eighteen hundred seven. Define $F(n,m)$ as the mth integer in the alphabetical list of the integers from one to n, and define $G(n,m)$ as the position of the integer m in this list. (a) What are the values of $F(n,m)$ and $G(n,m)$ when n = 1,000,000 and m = 1,000? When $n = m$ = 1,000,000? (b) What are the ten smallest values of n for which $F(n,n) = G(n,n) = n$?

—Allan Gottlieb's Puzzle Corner in *Technology Review*

Send your answers to any or all of the Summer Brain Ticklers to **Curt Gomulinski, Tau Beta Pi, P. O. Box 2697, Knoxville, TN 37901-2697** or e-mail to *BrainTicklers@tbp.org* as plain text only. The cutoff date for entries to the Summer column is the appearance of the Fall *Bent* in early October. The method of solution is not necessary, unless you think it will be of interest to the judges. We welcome any interesting problems that might be suitable for the column. The Computer Bonus is not graded. Curt will forward your entries to the judges who are **H.G. McIlvried, III,** *PA Γ '53,* **F.J. Tydeman,** *CA Δ '73,* **D.A. Dechman,** *TX A '57,* and the columnist for this issue, **J.C. Rasbold,** *OH A '83.*

The Ultimate 'Selfie'?

Our cover features spacewalk 'selfies' by NASA astronaut **Richard A. Mastracchio,** *Connecticut Beta '82,* who recently returned to Earth following a 188-day orbit aboard the International Space Station (ISS).

He has logged 228 days in space spanning four missions and served as a flight engineer for the latest, Expedition 38/39. Mastracchio is a veteran spacewalker and the 'selfies' were made during the three extravehicular activities (EVAs) he performed during his most recent mission. The first two were to remove and replace a faulty cooling pump, and the third to remove and replace a failed backup computer relay box.

Mastracchio stayed busy while aboard the ISS, including making a graduation ceremony speech to engineers at his alma mater, the University of Connecticut. Speaking from the ISS while orbiting Earth, he said, "I probably have the best job on and off the planet."