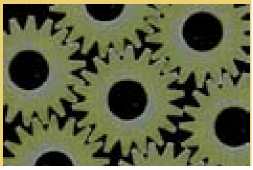


# Brain Ticklers

## RESULTS FROM WINTER 2006

**Perfect**

| | | | | |
|---|---|---|---|---|
| * | Brana-Mulero, Francisco J. | PR | A | '74 |
| | De Vincentis, Joseph W. | TX | Γ | '93 |
| * | Lints, Michael C. | NY | Π | '79 |
| * | Rasbold, J. Charles | OH | A | '83 |
| * | Schleehauf, Martin W. | NY | N | '79 |
| * | Spong, Robert N. | UT | A | '58 |
| * | Thaller, David B. | MA | B | '93 |
| * | Voellinger, Edward J. | non-member | | |
| | White, Adam D. | WI | A | '05 |

**Other**

| | | | | |
|---|---|---|---|---|
| | Aron, Gert | IA | B | '58 |
| * | Baines, Elliot "Chip" A., Jr. | NY | Δ | '78 |
| | Baldwin, Scott K. | MA | E | '91 |
| | Bennett, Ted | OH | I | '99 |
| | Biberdorf, Nathan | non-member | | |
| * | Bohdan, Timothy E. | IN | Γ | '85 |
| | Brule, John D. | MI | B | '49 |
| | Brzezinski, Mark A. | OH | B | '00 |
| * | Christenson, Ryan C. | UT | B | '93 |
| | Conway, David B. | TX | I | '79 |
| * | Couillard, J. Gregory | IL | A | '89 |
| | Couling, David | IN | Γ | '06 |
| | Doolittle, Scott | CA | AA | '07 |
| | Garnett, James M., III | MS | A | '65 |
| | Jones, Donlan F. | CA | Z | '52 |
| | Keesee, William R. | CA | A | '58 |
| * | Kimsey, David B. | AL | A | '71 |
| * | Klaver, David S. | NY | A | '05 |
| | Lalinsky, Mark A. | MI | Γ | '77 |
| | Lew, Thomas M. | TX | Δ | '84 |
| | Marrone, James D. | IN | A | '87 |
| * | McIlvain, Donald R. | PA | Δ | '52 |
| | Miller, Richard | MA | Z | '64 |
| | Mintz, Jeffrey A. | CA | M | '99 |
| | Mrozek, Paul M. | MI | H | '91 |
| | Nabutovsky, Joseph | Father of member | | |
| * | O'Reilly, Thomas | VA | B | '96 |
| * | Pecsvaradi, Thomas | PA | Z | '64 |
| | Petran, Eric | IN | Δ | '03 |
| | Quintana, Juan S. | OH | Θ | '62 |
| | Rentz, Peter E. | IN | A | '55 |
| | Robillard, David J. | MD | Γ | '88 |
| | Routh, Andre G. | FL | B | '89 |
| * | Schmidt, V. Hugo | WA | B | '51 |
| | Shepperd, Stanley W. | MA | B | '70 |
| * | Stribling, Jeffrey R. | CA | A | '92 |
| * | Strong, Michael D. | PA | A | '84 |
| * | Treacy, Stephen V. | NY | O | '89 |
| | Valko, Andrew G. | PA | Λ | '80 |
| * | Vogt, Jack C. | OH | E | '56 |
| | Wendling, D. Greg | IL | A | '78 |
| | Wheeler, William L. | NC | A | '84 |
| | Yee, David G. | NJ | B | '04 |
| | York, Jeff | NC | A | '85 |
| | Zison, Stanley W. | CA | Θ | '87 |

* Indicates correct bonus solution

## WINTER REVIEW

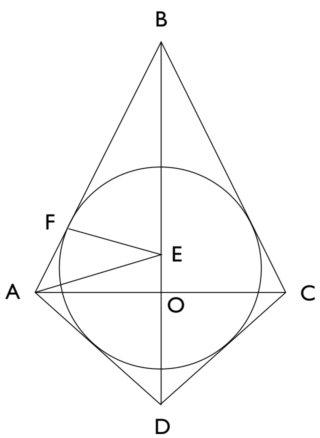
Problem No. 4, asking for the shortest ribbon to tie a box, proved to be very difficult, and it resulted in preventing 14 additional entries from being graded "Perfect."

## SPRING SOLUTIONS

Readers' entries for the Spring problems will be acknowledged in the Fall BENT. Meanwhile, here are the answers:

**1** The problem was to find a 10-digit number consisting of two each of 0s, 1s, 2s, 3s, and 4s, such that the 4s are separated by four digits, the 3s by three digits, etc. The smallest and largest such numbers are 1,214,230,043 and 4,131,243,200. You can obtain these with a manual trial and error process, starting at the left and always trying to insert the smallest/largest integer, respectively, as you build the sequence.

**2** Refer to the figure below. Let angle BAO = $\alpha$, angle DAO = $\beta$, angle EAO = $\gamma$, AE = $h$, OE = $d$, EF = $r$, and AO = OC = 1. Then, $d = \tan\gamma$, $h = 1/\cos\gamma$, and $r = h \sin(\alpha-\gamma)$. But, $\gamma = 0.5(\alpha+\beta)-\beta = 0.5(\alpha-\beta)$. Therefore, $r = \sin(\alpha-\gamma)/\cos\gamma = \sin[0.5(\alpha+\beta)]/\cos\gamma$. Letting $\alpha = 75^\circ$ and $\beta = 45^\circ$ gives $r = \cos 30^\circ/\cos 15^\circ$ and $d = \tan 15^\circ$. But $\cos 30^\circ = \sqrt{3}/2$ and $\cos 15^\circ = \sqrt{2+\sqrt{3}}/2$, so $r = \sqrt{6-3\sqrt{3}}$ and $d = (2-\sqrt{3})$.




**3** A will win if he gets four coins before B gets 6. The number of ways for A to win four coins while B wins $n$ coins (realizing that A must win the last coin) is $C(n+3, n)$, where $C(p, q)$ is the number of combinations of $p$ things taken $q$ at a time, with probability of $0.5^{n+4}$. Therefore, the probability of A winning is just the sum as $n$ goes from 0 to 5 of $C(n+3, n)0.5^{n+4}$ which equals $191/256 \approx 74.6\%$. The expected value, $E$, of A's winnings equals the sum of the products of his probability of winning or losing $x$ coins times $x$. Now, A can win up to 6 coins or lose up to 4. Thus, $E$ is the sum from 0 to 5 of $(6-n)C(n+3, n)0.5^{n+4}$ minus the sum from 0 to 3 of $(4-n)C(n+5, n)0.5^{n+6}$. This gives $E = 317/128 - 61/128 = 2$. In fact, at a point in a game where A has $x$ coins and B has $y$ coins, the expected value of A's winnings is $x - y$.

**4** Assume the following procedure. Add one $n$th of a liter from W to A, mix, and pour one $n$th of a liter from A down the drain. Repeat $n$ times. The original concentration of alcohol in A is 1. After the first transfer, the concentration is $C = 1/(1+1/n)$. After the second transfer, $C = [1/(1+1/n)] / [1+1/n] = 1/(1+1/n)^2$, or in general, after $n$ additions, $C = 1/(1+1/n)^n$. A few trials will show that $C$ decreases as $n$ increases. Therefore, $C_{min}$ equals the limit as $n$ approaches infinity of $1/(1+1/n)^n$, but this is just $1/e$.

**5** If the sculpture is stable when supported at any point on the hemispherical surface, it must be stable in a horizontal position with the fulcrum at the seam between the hemisphere and the cylinder. The volume of the hemisphere is $2\pi R^3/3$ and the volume of the cylinder is $\pi(R^2-r^2)L$, where $r$, $R$, and $L$ are the inside diameter, outside diameter, and length of the cylinder, respectively. Now the center of gravity of the hemisphere is at a distance of $3R/8$ from the center of the flat surface, and the center of gravity of the cylinder is at a distance of $L/2$ from the end. Therefore, if the sculpture is to be stable at any point on the hemispherical surface, we must have

$(2\pi R^3/3)(3R/8) = \pi(R^2 - r^2)(L^2/2)$. Solving for $L$ gives $L = R^2/\sqrt{[2(R^2 - r^2)]}$. Substituting $R = 50$ cm and $r = 49$ cm gives $L = 177.667$ cm. A little mathematics will show that the maximum distance from the far edge of the cylinder to the hemispherical surface is 234.57 cm.

**Bonus.** Consider a differential slice of diameter $x$ of the cone, a distance $y$ from the base. The probability of a random point being in that slice is given by $(\pi/4)x^2 dy/(1/3)(\pi D^2 h/4) = (3x^2/D^2 h)dy$, where $D$ is the diameter of the base and $h$ is the height of the cylinder. Also, $x/D = (h - y)/h$. Therefore, $x = D(1 - y/h)$. If this point is to be the closest point to the base of the cone, then the other points must be in the volume of the cone above $y$. This probability of a point's being above $y$ is: $x^2(h - y)/D^2 h = x^2(1 - y/h)/D^2 = (1 - y/h)^3$, and the probability of two points being above $y$ is $(1 - y/h)^6$. Therefore, the expected value of the distance of the closest point to the base is: $E = (9/D^2)\int_0^h x^2 y(1-y/h)^6 dy = (9/h) \int_0^h y(1-y/h)^8 dy$. Let $u = 1 - y/h$. Then, $y = h(1 - u)$ and $dy = -h du$. Making this substitution and integrating, we get: $E = 9h\int_0^1 u^8(1-u)du = 9h[u^9/9 - u^{10}/10]_0^1 = 9h(1/9 - 1/10) = h/10$.

**Double Bonus.** For the two calendars to be identical, January 1 must occur on the same day of the week, and the years must both be regular years or leap years. Julian years and Gregorian years will start on the same day of the week if the number of days difference between them is divisible by 7. Ten days were deleted in 1582, and the difference between the Julian calendar and the Gregorian calendar grows by a day each century year not divisible by 400. Therefore, the difference grew by one day in 1700, 1800, and 1900, making the Julian calendar 13 days different from the Gregorian calendar. The 14th day will be gained in 2100, which is a regular Gregorian year but a Julian leap year. Therefore, the first year they will agree is 2101. They will then stay in sync until 2200.

## NEW SUMMER PROBLEMS

**1** Although there are an infinite number of primes, we can find sequences of integers of any desired length, none of which are prime. Give an expression, in terms of $N$, for the first number of a string of at least $N$ consecutive composite integers. This expression does not have to identify the smallest such number.

—Daryl Cooper

**2** An archivist, baker, clergyman, and doctor played three rubbers of bridge (bridge is a four-person game with two partners on one side versus two partners on the other side). By coincidence, the score of each rubber was an exact multiple of 100. Each partnered each other once, and, at a penny a point, the doctor emerged $5 up. The clergyman and his brother lost the largest rubber by 800. They are a bit vague about other details, but the scoresheet shows that Peter finished $19 down, having fared better with Ronnie for partner than with Quentin. It shows too that Quentin did better overall than his father's only brother and that Sam did worse overall than the baker. The doctor, by the way, sent a greetings telegram when the archivist was born. And, also by the way, when Peter was Sam's current age, he could open beer bottles with his teeth. Who does what for a living?

—Martin Hollis

**3** Solve the following cryptic multiplication: WIT * WILL = THIRST with different letters representing different digits.

—*New Scientist*

**4** A point P lies in the same plane as a unit square. Let the vertices of the square, taken counterclockwise, be A, B, C, D. Also, let the distances from P to A, B, and C, be $u$, $v$, $w$, respectively. What is the greatest distance that P can be from D if $u^2 + v^2 = w^2$?

—34th Annual High-School Mathematics Examination

**5** Every pair of communities in a county is linked directly by exactly one mode of transportation: bus, train, or airplane. All three modes of transportation are used in the county with no community being serviced by all three modes and no three communities being linked pairwise (e.g., in a triangle) by the same mode. What is the maximum number of communities in the county? What is the maximum if only two modes of transportation are used and a community can be served by both modes? Besides the numbers, provide a labeled sketch of the networks.

—Tenth U.S.A. Mathematical Olympiad

**Bonus.** A hotel has rooms with cipher locks that open (indicated by a green LED) when the last four digits of any sequence keyed provide the correct combination. Thus, keying 01234 (five key strokes) tests both 0123 and 1234. If the combination is not known, what is the minimum number of keystrokes to guarantee that the lock can be opened?

—*Technology Review*

**Double Bonus.** Let $a(0) + b(0) + c(0) = 1$ with $0 \le a(0), b(0), c(0) < 1$ and $a(n+1) = a(n)^2 + 2b(n)c(n)$ with corresponding equations for $b(n+1)$ and $c(n+1)$. What is the limit of $a(n)$ as $n$ approaches infinity? Prove.

—G. R. Wachtell

Postal mail your answers to any or all of the Brain Ticklers to: **Jim Froula, Tau Beta Pi, P. O. Box 2697, Knoxville, TN 37901-2697** or email plain text to: *BrainTicklers@tbp.org*. The cutoff date for entries to the Summer column is the appearance of the Fall Bent in early-October. The method of solution is not necessary. We also welcome any interesting new problems that may be suitable for use in the column. The Double Bonus is not graded. Jim will forward your entries to the judges, who are: **Dr. H.G. McIlvried** III, *PA Γ '53*; **D.A. Dechman**, *TX A '57*; **J.L. Bradshaw**, *PA A '82*; and the columnist for this issue,

—**F.J. Tydeman**, *CA Δ '73*