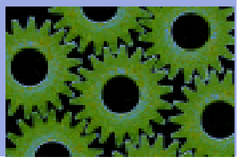


Brain Ticklers

RESULTS FROM WINTER 2004

Perfect

*	Bassler, Louis J.	AK	A	'82
*	Biggadike, Robert H.	AR	A	'58
	Couillard, J. Greg	IL	A	'89
	Galer, Craig K.	MI	A	'77
*	Garnett, James M., III	MS	A	'65
	Goodman, Thomas J.	PA	H	'05
*	Hess, Richard I.	CA	B	'62
*	Kay, Nathaniel E.	OH	Λ	'01
*	Mayer, Michael A.	IL	A	'89
	Pineault, Wayne	IL	Γ	'79
	Quintana, Juan S.	OH	Θ	'62
*	Schmidt, V. Hugo	WA	B	'51
*	Smith, Ronald E.	PA	A	'86
*	Verkuilen, William W.	WI	B	'92
*	Wegener, Stephen P.	LA	A	'75
*	Wolff, Nicholas L.	NE	A	'00
*	Zapor, Richard A.	CA	E	'84

Other

	Akridge, G. Russell	GA	A	'62
	Aron, Gert	IA	B	'59
	Bailey, Richard D.	MI	Γ	'80
*	Baines, Elliot A., Jr.	NY	Δ	'78
*	Bishop, Daniel J.	TX	Δ	'04
	Brule, John D.	MI	B	'49
	Conway, David B.	TX	I	'79
*	Doniger, Kenneth J.	CA	A	'77
*	Eiseman, Laurence H.	NY	Γ	'52
*	Griggs, James L., Jr.	OH	A	'56
	Heibeck, Georgine	Non-member		
*	Heiman, Daniel T.	NY	Γ	'03
	Johnson, Roger W.	MN	A	'79
	Jones, Donlan F.	CA	Z	'52
*	Kimsey, David B.	AL	A	'71
*	King, Joseph A.	GA	A	'04
	Koo, John S.	NY	P	'03
	Marks, Lawrence	NY	I	'81
	Marks, Ben	Son		
	Marrone, James D.	IN	A	'87
	Mazeika, Daniel F.	PA	B	'55
	McCormick, Michael J.	Son of member		
	Mrozek, Paul M.	MI	E	'91
*	Rasbold, J. Charles	OH	A	'83
*	Rentz, Peter E.	IN	A	'55
	Sedlak, Matthew	NY	Γ	'78
	Shurts, Garrick F.	MN	B	'04
	Otto, Cory	Non-member		
*	Snelling, William E.	GA	A	'79
	Snyder, M. Duane	IA	B	'63
	Dunne, Nick	Non-member		
*	Spong, Robert Neil	UT	A	'58
*	Stribling, Jeffrey R.	CA	A	'92
	Strong, Michael D.	PA	A	'84
	Thaller, David B.	MA	B	'93
*	Thompson, Davis R.	NY	A	'02
*	Treacy, Stephen V.	NY	O	'89
	Valko, Andrew G.	PA	Λ	'80
	Vinoski, Stephen B.	TN	Δ	'85
*	Voellinger, Edward J.	Non-member		
*	Vogt, Jack C.	OH	E	'56
*	Weinstein, Stephen A.	NY	Γ	'96
	Wong, Jimmy K.	CA	E	'94
*	Yee, David G.	NJ	B	'04

* denotes correct Bonus answer

WINTER REVIEW

Problem #1, which requested the largest integer such that each pair of consecutive digits is a different prime, was the most difficult. It prevented nine entrants from getting perfect ratings.

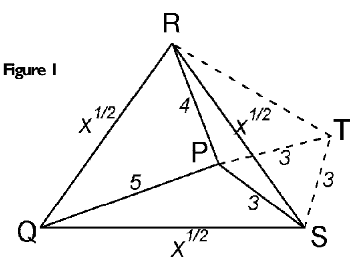
SPRING SOLUTIONS

Here are the solutions to the Spring 2004 Brain Ticklers. Spring entries will be acknowledged in the Fall.

1. Lance can prepare a set of 55 differently painted blocks for his daughter. There are three cases to consider: one set of opposite sides the same color, two sets of opposite sides the same color, and three sets of opposite sides the same color. In the first case, the color for the two opposite faces can be chosen in five ways. For each of these, the other four faces can be painted in 4!/(4x2) = 3 ways. (The factor of 4 eliminates rotations and the factor of 2 eliminates reflections.) Therefore, the total for the first case is 5 x 3 = 15. In the second case, the colors for the 2 sets of opposite sides can be chosen in C(5, 2) = 10 ways, where C(m, n) is the number of combinations of m things taken n at a time. The remaining two faces can be colored in C(3, 2) = 3 ways, for a total of 10 x 3 = 30. Finally, the three colors for painting the three sets of opposite sides in the third case can be chosen in C(5, 3) = 10 ways. Thus, the total is 15 + 30 + 10 = 55.

2. The two solutions for x that satisfy the set of four equations in the unknowns x, A, B, and C are $x = 25 \pm 12\sqrt{3}$. The problem is solvable using a numerical, algebraic, or geometric approach. For a numerical approach, choose values of A in 10º increments from 0º to 350º and sequentially calculate x, then B, then C. Then test to see if $A + B + C = 360º$. It turns out that $A = 150º$ and 330º are both solutions. An algebraic approach starts by reducing the four equations to two equations by eliminating x and C. This yields: $3 - 10\cos B + 8\cos A = 0$ and $7 + 30\cos B - 40\cos(A + B) = 0$. Next use $\cos(A + B) = \cos A\cos B - \sin A\sin B$ to replace the term $\cos(A + B)$. Then $\cos B$ can be eliminated from the two equations leaving one equation in $y = \cos A$: $96y^3 - 100y^2 - 72y + 75 = 0$. This cubic equation has three real roots: $y = 25/24$ and $y = \pm\sqrt{3}/2$. The first root is unphysical. The other roots yield $x = 25 \pm 12\sqrt{3}$.

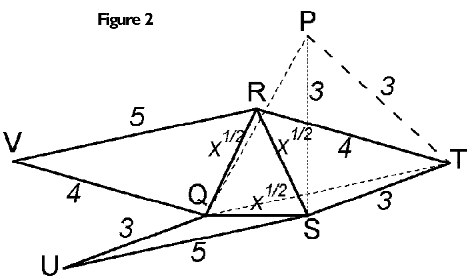
Figure 1



In the geometric approach, first rewrite the equations in the form $(x^{1/2})^2 = 3^2 + 4^2 - 2(3)(4)\cos A$, $(x^{1/2})^2 = 3^2 + 5^2 - 2(3)(5)\cos B$, and $(x^{1/2})^2 = 4^2 + 5^2 - 2(4)(5)\cos C$. These three equations represent the law of cosines for triangles with sides of $(3, 4, x^{1/2})$, $(3, 5, x^{1/2})$, and $(4, 5, x^{1/2})$. Thus, the triangles can be arranged to form equilateral triangles with sides of length $x^{1/2}$.

In *Figure 1*, note that $A + B + C = 2\pi$ is identically satisfied about the point P. To solve for x construct the equilateral triangle PST. Now PSQ = π/3 – PSR = RST. Therefore, ΔRST is congruent to ΔPQS, which means that RT = 5, so RPT = π/2, and RPS = A = π/3 + π/2 = 5π/6. Therefore, x = 25 – 24cos(5π/6) = 25 + 12√3.

Figure 2



Diagrams such as *Figure 2*, on the other hand, can be drawn for any value of $x < 49$; x is uniquely determined only when the additional condition $A + B + C = 2\pi$ is imposed. In *Figure 2*, construct the equilateral triangle SPT, and draw line segments PQ and QT. Note that QSP = π/3 + RSP and RST = π/3 + RSP. Thus, QSP = RST and ΔQSP is congruent to ΔRST. Therefore, PQ=4. Now note that A = 2π - STR, B = STQ, and C = QVR. Finally, QTR + STQ = STR = 2π – A, or QTR + B + A = 2π. Thus, the condition $A + B + C = 2\pi$ will be satisfied only when QTR = C, making ΔVQR congruent to ΔTQR. Then QT = 5, QPT = π/2, and QPS = STR = π/2 - π/3 = π/6, and A = 2π– STR = 2π - π/6 = 11π/6 so that x = 25 – 24cos(11π/6) = 25 - 12√3.

3. Ann's scores for the four rounds of the golf tournament were 61, 67, 71, 79; Betty's scores were 62, 65, 74, 77; Carol's scores were 63, 68, 72, 75; Donna's scores were 64, 66, 70, 78. One logical approach would proceed as follows: the 20 possible scores for the golfers are the integers 60 through 79. Of these, 61, 67, 71, 73, and 79 are prime, and 62, 65, 69, 74, and 77 are semiprime. Ann's scores must be four of the five primes, and Betty's scores must be four of the five semiprimes. Because the primes sum to 351, and the semiprimes sum to 347, the unused prime must be four more than the unused semiprime. Thus, the numbers not used are 73 and 69 and each golfer's score for the four rounds was 278.

Now, the sum of the ten possible scores that are neither prime nor semiprime is 692. Of these, the sum of the values not used must be 692 – 2(278) = 136. The only ways to eliminate 136 are to not use (60, 76), (64, 72), or (66, 70). Because there are only two odd numbers among the possible scores for Carol and Donna, they must have been scored by the same golfer. From this point, trial and error quickly shows that the only possible solution is as stated above.

4. The maximum number of intersections when m points of one line are connected by straight line segments to n points of a second, parallel line are $mn(m – 1)(n – 1)/4$. Draw the two lines horizontally with the upper line containing n points and the lower line containing m points. Consider the left endpoint on the lower line. When this point is connected to the n points on the upper line, no intersections result. Now consider the point next to the left endpoint of the lower line. When it is connected to the n points (counting in order from left to right), the first line segment intersects $n – 1$ lines, the second line segment intersects $n – 2$ lines, the third $n – 3$ lines, etc., until the nth line intersects $n – n = 0$ lines, for a total of $n(n – 1)/2$ intersections.

Consider the next point on the lower line. The first line will cross $2n – 2$ lines, the second line will cross $2n – 4$ lines, etc., for a total of $2[n(n – 1)/2] = n(n – 1)$. Similarly, the lines from the fourth point will result in $3n(n – 1)/2$ intersections.

In general, connecting the ith point to the n points will produce $(i – 1)n(n – 1)/2$ intersections. Therefore, the total number of intersections is $n(n-1)(0 + 1 + 2 + 3 + … + m – 1)/2 = mn(m – 1)(n – 1)/4$.

5. Lou scored 36 goals last season, Mac scored 26, and Neil scored 11. From the problem statement, we know that QR = PQ – QP = 10P + Q – 10Q – P = 9(P – Q). This means that the only possible values for QR are 18, 27, 36, and 45 (Q must be less than P since PQ > QP). Also, we know that RS = QR + SR, or QR = RS – SR = 9(R – S). For each possible QR, we can use these relationships to calculate the corresponding values of R and S. In this manner we find that the only possible values for (QR, P, S) are (18, 3, 6), (27, 5, 4), (36, 7, 2), and (45, 9, 0). Since PQ > RS and S is not zero, RS must be 62, and (P, Q) = (7, 3).

Bonus. When $x = n(2n+1)$, then the sum of the squares of the first $n+1$ integers in the sequence $x, x+1, x+2,…$ equals the sum of the squares of the next n terms in that sequence. Stated using summation notation, we want a formula for x such that

$\sum_{i=0}^{n}(x+i)^2 = \sum_{i=n+1}^{2n}(x+i)^2$. Because $(x + i)^2 = x^2 + 2ix + i^2$, we have

$$\sum_{i=0}^{n}(x+i)^2 = x^2\sum_{i=0}^{n}1 + 2x\sum_{i=0}^{n}i + \sum_{i=0}^{n}i^2 = (n+1)x^2 + n(n+1)x + \frac{n(n+1)(2n+1)}{6}.$$

Similarly,

$$\sum_{i=n+1}^{2n}(x+i)^2 = nx^2 + n(3n+1)x + \frac{n(2n+1)(4n+1)}{3} - \frac{n(n+1)(2n+1)}{6}.$$

Equating the rightmost sides of these two equations and rearranging gives $x^2 – 2n^2x – n^2(2n+1) = 0$, which can be factored to give $(x + n)(x – 2n^2 – n) = 0$. Since x is positive, this means $x = 2n^2 + n = n(2n+1)$.

Computer Bonus. One 4 x 4 magic square with elements consisting of prime integers less than 100 and a magic sum of 202 is

11	5	89	97
83	79	17	23
67	47	59	29
41	71	37	53

This answer is one of 32 equivalent magic squares. Eight equivalent squares are produced by rotations and reflections. For each of these, another square can be formed by interchanging columns 1 and 2 and columns 3 and 4, followed by interchanging rows 1 and 2 and rows 3 and 4. Then, from these 16 squares, 16 more squares can be formed by interchanging columns 2 and 3 followed by interchanging rows 2 and 3. Thus, it is always possible to arrange the square so that the smallest number on a diagonal is in the upper left corner and the number next to it on the first row is smaller than the number below it in the first column. We call this arrangement a primitive solution. Considering all possible magic sums for 4 x 4 magic squares with prime factors less than 100, we found 1,502 primitive solutions. However, the answer above is the only primitive solution with a magic sum of 202.

NEW SUMMER PROBLEMS

1. Reproduce the multiplication problem shown below with all of the missing digits properly inserted.

```
      * * 7 *
        * 7 *
    ---------
    * * * * *
 * * * 2 *
 8 * 5 *
 -------------
  * * * * * *
```

—*Mathematical Puzzles for Beginners & Enthusiasts*
by Geoffrey Mott-Smith

2. At the post office in Stamptopia, one can buy any denomination of stamp from 1¢ to $1 inclusive. Three couples came to buy stamps. All the stamps bought by each individual were of the same denomination. The number each purchased equaled the value of one of the stamps he or she bought. Paul bought 17 more stamps than Sue, and Quentin bought 7 more stamps than Tess. Each husband spent 45 cents more than his wife. Who is married to whom? (The names of the other two people involved are Ron and Ursula).

—*You are a Mathematician*
by David Wells

3. Consider a square with unit sides. Now draw a circle with radius r and center coincident with the center of the square. What value of r will minimize the area included in either the square or the circle but not common to both?

—Southwest Missouri State
University Dept. of Mathematics

4. When a certain radar installation detects an approaching plane, it sends out a signal. If the plane responds with the correct code, no action is taken. If, however, a correct response is not received, a missile is launched. To ensure that friendly aircraft are not destroyed, the radar site sends its signal on five channels. If correct replies are received on at least three of these, no action is taken. If p is the probability that a channel fails, for whatever reason, what is the value of p so that there is only one chance in a million of firing on a friendly aircraft? Assume p is the same for all channels.

—Adapted from
Duelling Idiots and other Probability Puzzlers
by Paul J. Nahin

5. After an operation to remove his cancerous thyroid, John swallowed a radioactive sodium-iodide solution containing 150 millicuries of iodine-131 to destroy any remaining thyroid cells. Immediately thereafter (at 10 a.m.), a Geiger counter showed a radiation level of 190 millirems. John cannot be released from isolation until his radiation level is 5 millirems or less. Radiation is reduced by two mechanisms: radioactive decay and excretion in bodily fluids. Typical half -retention time for patients is between 0.75 and 1.0 days, and the half life of iodine-131 is 8.05 days. If only one Geiger counter measurement is made each day (at 10 a.m.), what is the range of release dates John was initially told? John's actual half-retention time was 0.84 days. What was the release date, and what was the Geiger counter reading at 10 a.m. on that day?

—**Ronald L. Hartman,** *LA* Α *'55*

Bonus. Assume the hands of an accurate analog twelve-hour clock coincide at 12:00. When will the hour, minute, and second hands most nearly trisect the clock's face, that is, have angles as close to 120 degrees between pairs of hands as possible? In other words, if the three angles are A, B, and C, find when, during a twelve-hour period, the function $|A - 120| + |B - 120| + |C - 120|$ is a minimum, where $|x|$ means the absolute value of x.

—G.R. Williams

Computer Bonus. A weak prime is a prime such than when any of its digits is replaced (one at a time) by any different digit, the resulting integer is no longer prime. The smallest weak prime is 294,001. Find the next smallest weak prime.

—*The Prime Puzzles & Problems Connection*

Send your answers to any or all the Summer Brain Ticklers to:

Jim Froula, Tau Beta Pi
P.O. Box 2697
Knoxville, TN 37901-2697

The cutoff date is the appearance of the Fall issue in mid-October, and acknowledgments will be made in the Winter '05 column. If your answers are text only, they may be e-mailed to *Brainticklers@tbp.org.* The details of your calculations are not required, and the Computer Bonus is not graded. Jim will forward your entries to the judges:

H.G. McIlvried III, *PA* Γ *'53,*
D.A. Dechman, *TX* Α *'57,*
F.J. Tydeman, *CA* Δ *'73,*

and the columnist for this issue,

—**J.L. Bradshaw,** *PA* Α *'82.*