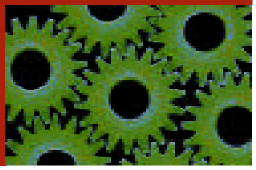

SPRING ANSWERS

Here are the solutions for the Spring Brain Ticklers. Entries will be acknowledged in the next issue.

1 The volume of a sphere is equal to $4\pi r^3/3$, and the volume of a spherical segment is equal to $\pi h^2(3r-h)/3$, where r is the radius of the sphere and h is the height of the segment. For the segment to be half the volume of a hemisphere, we have $\pi h^2(3r-h)/3 = \pi r^3/3$, or $3rh^2-h^3 = r^3$. Rearranging gives $(h/r)^3-3(h/r)^2+1 = 0$, a cubic equation which can be solved analytically to give $h/r = 1-2\sin(\pi/18)$. For $r = 1$, $h = 1-2\sin(\pi/18) = 0.6527$ m, which is the depth of the water when the tub is half full.

2 Clearly, $f(9) = 1$. Now, for $f(99)$ there are 10 ones for the tens-digits from 10 to 19 plus 1 one for each 10 integers from 0 to 99. Therefore, $f(99) = 10+10f(9) = 20$. Similarly, $f(999) = 100+10f(99) = 300$; or in general, $f(10^n-1) = 10^{n-1}+10f(10^{n-1}-1) = n10^{n-1}$. Also, $f(20) = 10+2f(9) = 12$; $f(200) = 100+2f(99) = 140$; or, in general, $f(2\times10^n) = 10^n+2f(10^n-1) = 10^n+2n10^{n-1}$.

From this last formula, we see that $f(200{,}000) = 200{,}000$, but this is not the smallest n. If we work our way backward, we see that each integer between 199,982 and 199,999 contributes 1 to the function, except 199,991, which contributes 2. Therefore, we have $f(199{,}981) = 199{,}981$.

Notice that 199,981 also contributes 2 to the function, so that $f(199{,}980) = 199{,}979$.

Furthermore, all the numbers from 100,000 to 199,980 contribute at least 1. Also, $f(99{,}999) = 50{,}000$ and $f(20{,}000) = 18{,}000$, so there can be no numbers greater than 1 and less than 199,981 for which $f(n) = n$.

3 The "clues" for the date are that it is (1) odd, (2) greater than 13, (3) not square, (4) a cube, and (5) less than 17. Since (2) and (5) cannot both be false and since only one clue is true, either (2) is true or (5) is true. Consequently, (1), (3), and (4) are all false, which means that the date is even, a square and not a cube. Thus, it must be 4 or 16, but it cannot be 16, since that would make both (2) and (5) true. Therefore, the date is July 4.

4 Divide DEEPDEEP by DEEP to give 10,001, which has prime factors of 73 and 137. Therefore, SINK must be a multiple of one of these factors, and THEM must be a multiple of the other. It is then easily determined by a simple computer program (or a bit more tediously with a hand calculator) that the solution is 5069×2847 = 14431443 for SINK×THEM = DEEPDEEP.

5 Assume the eighth red ball is drawn on the nth draw, leaving $80-n$ balls in the urn. The ways the n-1 preceding balls can be drawn is $C(n-1,7)$, where $C(i,j)$ is the combinations of i things taken j at a time, which is also the permutations of i things, where j of them are identical and the other $i-j$ are also identical. Now, the total ways the 80 balls can be drawn is $C(80,8)$. Therefore, $E=$ $[72C(7,7)+71C(8,7)+70C(9,7)+...+0C(79,7)]/C(80,8)$.

Making use of the number theory identity $C(a,a)+C(a+1,a)+C(a+2,a)+...+C(a+k,a) = C(a+k+1,a+1)$ repeatedly gives $E = [C(8,8)+C(9,8)+C(10,8)+...+C(79,8)]/C(80,8)$.

Applying the identity once more gives $E = C(80,9)/C(80,8)$. But, $C(80,9) = 8C(80,8)$. Therefore, $E = 8$. This problem can also be solved using a probability tree.

Bonus. We can calculate the time and velocity of the ball as it just enters the water from the relationships $v = \sqrt{2ah}$ and $t_1 = v/a$, where $h = 2$ m, and $a = 9.8$ m/s^2. Therefore, $t_1 = 0.63888$ s, and $v = 6.26099$ m/s. In the water, since $kA = 1.0$, we have $F = ma = m\,dv/dt = v^2+Q\rho_w g-mg$, where $Q = 0.000113097$ m^3, $\rho_w = 1{,}000$ kg/m^3, and $m = 0.112$ kg. In the equation for F, v^2 represents the drag force, $Q\rho_w g$ represents the buoyancy force, and mg represents the gravitational force. Therefore, $dt = m\,dv/(v^2+b^2)$, where $b = \sqrt{Q\rho_w g-mg} = 0.10370$. Integrating gives $t_2 = (m/b)\tan^{-1}(v/b)$, or $v = dh/dt = b\tan(bt/m)$, which can be integrated to give $h = -m\ln[\cos(bt/m)]$. Substituting values gives $t_2 = 1.67863$ s and $h = 0.45926$ m.

On the way back to the surface, the force equation yields $m\,dv/dt = b^2 - v^2$. Integration gives $t_3 = (m/2b)\ln[(b+v)/(b-v)]$. Solving for v gives $v = dh/dt = b[1-2/(e^{2bt/m}+1)]$. A second integration gives $h = m\ln[(e^{2bt/m}+1)/2]-bt$. This can be rearranged to produce $e^{2bt/m}- 2e^{h/m}e^{bt/m}+1 = 0$, which is a quadratic in $e^{bt/m}$. Letting $h = 0.45926-0.03 = 0.42926$ m, and solving for t gives t_3 = 4.88794 s.

Therefore, the total elapsed time from the instant the ball leaves the boy's hand until it just breaks the surface upon reappearing is $t_1+t_2+t_3$ = 7.2054s.

Computer Bonus. The sum σ of the prime factors of any integer $N = p_1^{a_1} p_2^{a_2} \ldots p_n^{a_n}$ is given by the number theory equation

$$\sigma(N) = \frac{p_1^{a_1+1} - 1}{p_1 - 1} \bullet \frac{p_2^{a_2+1} - 1}{p_2 - 1} \bullet \ldots \bullet \frac{p_n^{a_n+1} - 1}{p_n - 1}$$

The approach is to write a program to calculate successive squares, compute their σ's, and test to see if σ is a cube. The smallest integer for which $\sigma(k^2) = m^3$ is found to be k = 43,098, for which we have $k^2 = 2^2 3^2 11^2 653^2$. Applying the above formula, we find that $\sigma(43{,}098^2)$ = 5,168,743,489 = $1{,}729^3$.

NEW SUMMER PROBLEMS

1 Mr. Meek is pleased with his new extension phone number because it has four digits, the middle two of which are identical. "Like my name," he explains. The repeated digit is also the first digit of Mr. Humble's new four-digit number. Moreover, Meek's first digit is the same as the first digit of Mr. Lowly's new four-digit number.

If you interchange the first and last digits of Lowly's number, you get Humble's. If you subtract Lowly's number from Humble's, you get Meek's. So, what is Meek's new number?

— *New Scientist*

2 I know that the lifetime batting averages of Mike and John are 0.200 and 0.300, but I forget which is which. However, in Mike's first 10 times at bat this year he had three hits, and in John's first 10 times at bat this year he had two. What is the probability that Mike's lifetime batting average is 0.300? (Assume that the probability that a batter will get a hit is equal to his lifetime batting average, which is the ratio of number of hits to number of times at bat.)

— **Byron R. Adams,** *TX* Α *'58*

3 Solve this cryptic addition:

```
   OLD
  SALT
  TOLD
- TALL
 TALES
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— *Challenging Mathematical Teasers* by J.A.H. Hunter

4 Sue has opened her piggy bank, which contains a mixture of pennies, nickels, dimes, and quarters. She quickly notes that she has at least four of each kind. Upon careful counting, she finds that she has exactly 100 coins with a face value of $2.73. How many of each kind does she have?

— **Don A. Dechman,** *TX* Α *'57*

5 One good way to estimate the height of an object is to take a known height, sight along a ruler until the known object subtends an easy length to work with, and then take the proportional height subtended by the unknown object. If the ruler is parallel to both objects, the result will be exact.

Recently I attempted to measure the height of a bridge this way. There is a plaque on the side of the vertical bridge tower marking the water level from the last flood — three meters from the ground. I backed off and held my ruler so that the plaque's height appeared to be three centimeters. The height of the roadway then appeared to be 12 cm and that of the supporting tower 30 cm.

Hence, I estimated that the roadway is 12 m above ground and that the tower is 30 m high. Later I found that the roadway is actually 13 m above the ground. Clearly, the discrepancy arose because I did not hold the ruler precisely vertical. What, then, is the correct height of the tower?

— Adapted from *Technology Review*

Bonus. As I was jogging in the predawn darkness, Venus was the brightest object in the eastern sky, and I wondered if it were ever brighter. I found that Venus and the Earth are in nearly circular coplanar orbits, with an orbital radii ratio of 0.723. Assume that the sunlight reflected by Venus towards Earth is proportional to the illuminated fraction of the Venus disk as seen from Earth. Then, at what value of the Earth-Sun-Venus angle does Venus appear brightest? Give your answer accurate to the nearest 0.1 degree.

— **James L. Griggs Jr.,** *OH* Α *'56*

Computer Bonus. Find the smallest and largest hexadecimal integers such that each of their squares contains each of the 16 hexadecimal digits (0 through 9 and A through F) once and only once. A leading zero is not allowed.

— **Howard G. McIlvried III,** *PA* Γ *'53*

The judges are: H.G. McIlvried III, *PA* Γ *'53;* F.J. Tydeman, *CA* Δ *'73;* D.A. Dechman, *TX* Α *'57;* and the columnist for this issue,

— **R. Wilson Rowland,** *MD* Β *'51.*