Brain Ticklers

RESULTS FROM FALL

Perfect

*Bohdan, Timothy E.	IN	Γ	’85
Couillard, J. Gregory	IL	Α	’89
Gerken, Gary M.	CA	Η	’11
*Griggs Jr., James L.	OH	Α	’56
Richards, John R.	NJ	Β	’76
*Slegel, Timothy J.	PA	Α	’80

Other

Alexander, Jay A.	IL	Γ	’86
Aron, Gert	IA	Β	’58
Budd, Christopher M.	AZ	Β	’94
Dechman, Don A.	TX	Α	’57
Ellis Jr., Ira T.	PA	Η	’56
*Gaston, Charles A.	PA	Β	’61
Gee, Albert	CA	Α	’79
Gee, Aaron J.	CA	Ψ	’16
*Gulian, Franklin J.	DE	Α	’83
Gulian, William F.	Son of member		
Handley, Vernon K.	GA	Α	’86
Johnson, Mark C.	IL	Α	’00
Johnson, Roger W.	MN	Α	’79
Jordan, R. Jeffrey	OK	Γ	’00
Kimsey, David B.	AL	Α	’71
Lalinsky, Mark A.	MI	Γ	’77
Marrone, James I.	IN	Α	’61
*Norris, Thomas G.	OK	Α	’56
Norris Jr., Thomas G.	PA	Γ	’79
Rentz, Peter E.	IN	Α	’55
Riedesel, Jeremy M.	OH	Β	’96
Schmidt, V. Hugo	WA	Β	’51
Spong, Robert N.	UT	Α	’58
Summerfield, Steven L.	MO	Γ	’85
Voellinger, Edward J.	Non-member		
Zison, Stanley W.	CA	Θ	’87

*Denotes correct bonus solution

FALL REVIEW

The Fall Ticklers appear to have been more difficult than usual. Fewer than half the answers submitted for problems 2 (Hook, Line, and Sinker), and 4 (queens on a chess board) were correct. Also, only half the solutions submitted for the Bonus problem (numbers on foreheads) were correct.

We apologize to **Hubert W. Hagadorn,** *PA E ’59,* for misspelling his name as the submitter of Winter Problem No. 4.

WINTER ANSWERS

1 CORNELL + STUDENT = ENDURES translates into **5481977 + 3620916 = 9102893.** For there not to be a carry into the fourth column (denoted here as a binary flag c_4), E<5 leading either to $L+N+c_5<5$ or $9<L+N+c_5<15$. For the first case, L≠0 because this implies T=S. N≠0 because this implies D=U under the assumption of no c_4. L and N cannot take on the pair of values 1 and 2 (nor 1 and 3) in either order because this would eliminate all valid possibilities for C and S. So $L+N+c_5>=5$. For the second case, there must be a carry c_5, which implies a carry c_6. Checking where E=3 and L+N=12 yields no combination of L and N (there are four) resulting in a valid set of S,C,T,R,O,U and D. Similar results are found when E=4 and L+N=13 (again, in four combinations). So c_4 must exist.

Note $1+N+D=10c_3+U$ and $R+U+c_3 = D+10c_2$. Combining these constraints yields $R+N = 9c_3-1+10c_2$. It follows that c_3 or c_2 must be a carry but not both, and R+N = 8 or 9.

Now consider the carry c_5, which would arise only if there is a carry c_6 and L+N>13, or if there is no carry c_6 and L+N>14. Considering values of L (which cannot be 9) and N that satisfy this constraint, and the resulting values of E and R, only one pair obeys the rule that R+N = 8 or 9, which is L=6, N=8 implying there is a carry c_6 leading to E=5 and R=1. It follows that S=3, C=2, T=7, and O=0. Alas, the remaining letters U and D cannot be filled with 4 and 9 in any valid combination, and there can be no carry c_5. This fact creates the restriction 4<L+N<9.

Under this last constraint, take first the case where there is no carry c_6. Considering values of L and N that satisfy this constraint, and with the resulting values of E and R, only two sets of these letters obey the rule that R+N = 8 or 9.

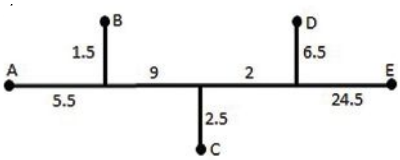
The first has L=2, N=5, E=7 and R=4, leading to T=1 and S=3, but no valid choice for C. The second has L=5, N=3, E=8 and R=6, leading to T=2, S=7 and C=1, and O=0 with a necessary carry c_2. Alas, once again the remaining letters U and D cannot be filled with 4 and 9 in any valid combination, and there must be a carry c_6. In this case, we once again consider values of L and N that satisfy the above constraint, and with the resulting values of E and R, only four sets of these letters obey the rule that R+N = 8 or 9.

The first has L=1, N=5, E=7 and R=4. It implies that T=9, but leads to no valid choice for S. The second has L=5, N=2, E=8 and R=6, leading to T=9, S=3 and C=4, but no valid choice for O. The third has L=4, N=3, E=8 and R=6, leading to T=7, S=2 and C=5, but no valid choice for O. This forces L=7, N=1, E=9 and R=8. T cannot be 4, because S cannot be 1. T cannot be 5, because it leads to S=2, C=6 and no valid choice for O. Thus T=6, S=3, C=5, and O=4. The problem is fully solved by assigning D=0 and U=2.

2 **The order of the months in Bonkers’ year is Collywobble, Stephen, Strawberry, Jinks, Pinafore, Tiddlywinks, and Cucumber.** In 21 a.b., **Collywobble 2nd is a Joyday.** With 10 weeks (50 days) between Collywobble 41st and Pinafore 7th, Collywobble accounts for 3 days and Pinafore 7. This leaves 40 days to account for by the middle months. The only way to achieve 40 days is to have a 19-day month, a 17-day month, and a 4-day month in some order. Strawberry or Cucumber is one of the months, Tiddlywinks or Jinks is another of the months, and Stephen is a third. Since Blissday is two days after Funday, the king’s holiday was 7×5+2=37 days long. Strawberry accounts for 2 days, and Tiddlywinks accounts for 8. There are, therefore, 27 days to account for by middle months. There are two situations to consider here: the first arises when the king returns sometime after the new year in 18 a.b. In this case, Collywobble would contribute 13 days, leaving 14 days unaccounted for. The only way to partition what is left is to include Pinafore (10) and Stephen (4), the latter of which must be after Collywobble in 18 a.b. We know, however, that Pinafore must be four months after Collywobble, making the above arrangement impossible; the king must return sometime in 17 a.b. In this situation, the only way to achieve 27 days is with Pinafore and Jinks in some order. Collywobble cannot be between Strawberry and Tiddlywinks (cyclically speaking), so must be either right after Tiddlywinks or the following month. If it is immediately after Tiddlywinks, Pinafore must be right after Strawberry, and Cucumber would need to follow Pinafore, but since this is between Strawberry and Tiddlywinks, this is a contradiction. So, Collywobble is two months after Tiddlywinks, Pinafore is just before Tiddlywinks, and Cucumber is just after Tiddlywinks. It follows that Stephen must be directly after Collywobble and Jinks directly after Strawberry. In 21 a.b., Collywobble has 44 days and Stephen has 4 days. There are 42+4+13=59 days between the dates in question. Since 59mod5=4, Strawberry 13th (a Workday) is four weekdays after Collywobble 2nd, making the latter a Joyday.

3 The x with the smallest positive numerator and denominator* is **1681/144.** Let x be represented as p^2/q^2, so the remaining two numbers of interest are $(p^2 \pm 5q^2)/q^2$. Considering just the numerators, define $a^2=p^2-5q^2$, $b^2=p^2$, and $c^2=p^2+5q^2$. Then, define $D=c^2-b^2=b^2-a^2$. Rearranging gives two equations $c^2=b^2+D$ and $a^2=b^2-D$. Multiplying these two equations together and rearranging gives $(ca)^2+D^2=(b^2)^2$. This is a well known Diophantine equation of the same form as is used for generating Pythagorean triples. It has a parametric solution for integer m and n: $ca = m^2-n^2$, $D=2mn$, $b^2=m^2+n^2$, the latter expression being of a similar Diophantine form. Repeating, if for integer r and s we set $b=r^2+s^2$, $m=2rs$, $n=r^2-s^2$ and plug them into the expressions for D and b^2 we get $D=4rs(r^2-s^2)$ and $b^2=(r^2+s^2)^2$. It remains to find integers r and s that make the smallest integer of the form $5q^2$. Either r or s must be 5, and a quick check reveals r=5 and s=4 give $D=720=5\times12^2$ and $q^2 = 144$. $b^2=(5^2+4^2)^2=p^2=1681$, so x=1681/144. [* **James R. Roche**, *IN Γ '85,* rightly points out there is technically no "smallest x" when measuring the value of *p/q:* provided *p* and *q* are sufficiently large *p/q* can get arbitrarily close to 0. Despite the misleading wording, our 13th century mathematicians were indeed angling for the simplistic ratio with smallest positive integer *p* and *q*—the contest's answer given above was found by none other than Fibonacci!]

4 The number of towns is **five,** and they are **connected as depicted in the diagram below**. Total road length is 51.5km.



The number of towns must be five, since the number of routes is 10 and C(10,2) = 5. Since there is a unique path between each pair of cities, there can be no loops. For 5 towns, the above picture shows the generic topology—note however that other derivative forms could be created by setting individual road segments to a length of zero. Assigning the seven road segments with variables a through g, we can find the route length between any of the cities A,B,C,D or E by the following ten equations (AB denotes the distance from A to B):

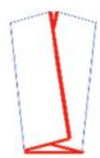
1. AB = $a + b$
2. AC = $a + c + d$
3. AD = $a + c + e + f$
4. AE = $a + c + e + g$
5. BC = $b + c + d$
6. BD = $b + c + e + f$
7. BE = $b + c + e + g$
8. CD = $d + e + f$
9. CE = $d + e + g$
10. DE = $f + g$

To solving for the road segment lengths, a and b are obtained from (1) and the difference of (2) and (5); f and g are obtained from (10) and the difference of (8) and (9); c and e are obtained from (3) and the difference of (2) and (8); d is obtained from (2). The resulting equations are:

a = (AB + AC – BC)/2
b = (AB – AC + BC)/2
c = (AD – CD – AB + BC)/2
d = (AC – AD + CD)/2
e = (AD – AC – DE + CE)/2
f = (DE + CD – CE)/2
g = (DE – CD + CE)/2

One approach to solving is to write a quick program: start from the first ten primes, choose seven and assign them to the various route lengths. With the resulting a through g, check to ensure that the remaining three routes are equal in length to the remaining three primes. In the event no valid solution is found, increase the window of primes by one and repeat. Doing so quickly reveals that a valid solution exists with primes from 7 through 41 and has a=5.5 km, b=1.5 km, c=9 km, d=2.5 km, e=2 km, f=6.5 km and g=24.5 km (resulting in a total road length of 51 km). Additionally, AB= 7 km, AC=17 km, AD=23 km, AE=41 km, BC=13 km, BD=19 km, BE=37 km, CD=11 km, CE= 29 km and DE=31 km. Note that symmetrical equivalent solutions will exist where a, b, f and g can be interchanged, and c and e can be interchanged.

5 **Five folds** are required. One solution follows: Create a horizontal fold by bringing the top edge to the center of the square. Next bring the right edge in, making a not-quite vertical fold that has a steep positive slope. The uppermost point after the fold will be just to the right of the center vertical line of the original square. Repeat in a symmetric way with the left edge; this results in the figure at left:

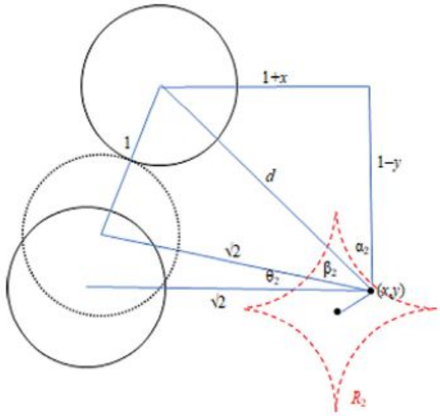


Next create a horizontal fold about ⅓ of the way down from the top, and finish by taking the bottom portion, folding it up and into the pocket created by the previous fold. The result will look something like this:

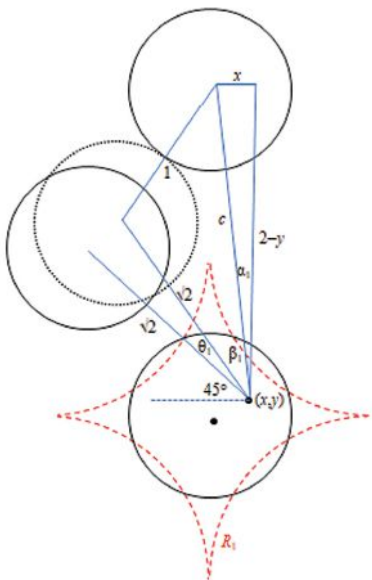
Bonus The probability that no overlap will occur is **approximately 2.49%.** To approach this problem, it may help to create a physical model with a grid of polka dots printed on paper and 4 additional dots in the same pattern placed on a plastic transparency that can be moved around. A little sleuthing will show that, to avoid overlapping dots, the center of the plastic square can only be in two regions. The first region, R_1, is centered around the center of a polka dot on the tablecloth (with the plastic not rotated with respect to the grid). The second region, R_2, is centered about the center of the square defined by the center of four adjacent polka dots (with the plastic rotated by 45° with respect to the grid). By shifting the plastic about the centerpoint to see the maximum extent that can be reached before overlap occurs, it is readily apparent that each region is vaguely diamond shaped, where the 4 convex "sides" are arcs of circles. In R_1 the circle radius is 1, and in R_2 the circle radius is $r=1-\sqrt{(2\sqrt{2}-2)}$. When the plastic is translated from the region R_i center by (x,y), we wish to find out the amount of rotation θ_i about (x,y) that is allowed before the polka dots on the plastic begin to overlap the polka dots on the tablecloth. Symmetry enables us to focus on one quadrant of the region, a clockwise rotation, and the behavior of one dot on the plastic. Out of $\pi/2$ radians, there will be $2\theta_i$ radians of rotation with no overlap, or a $4\theta_i/\pi$ probability. The overall probability will then be

$$\frac{4}{\pi}\int_0^1\int_0^{1-\sqrt{1-(x-1)^2}}\theta_1(x,y)\,dy\,dx$$
$$+\frac{4}{\pi}\int_0^r\int_0^{r-\sqrt{r^2-(x-r)^2}}\theta_2(x,y)\,dy\,dx$$

It remains to find $\theta_i(x,y)$. The following diagram depicts the case around R_1:



$c^2=x^2+y^2-4y+4$, and $\alpha_1=\text{acos}((2-y)/c)$. Using the law of cosines gives $\beta_1=\text{acos}((1+c^2)/(2\sqrt{2}c)$. We arrive at $\theta_1=\pi/4-\alpha_1-\beta_1=\pi/4-\text{acos}((2-y)/\sqrt{(x^2+y^2-4y+4)})-\text{acos}((x^2+y^2-4y+5)/(2\sqrt{2}\sqrt{(x^2+y^2-4y+4)}))$. The diagram below depicts the case around R_2 (note it is not to scale due to the small size of the region):



$d^2=x^2+y^2+2x-2y+2$, and $\alpha_2=\text{acos}((1-y)/d)$. Using the law of cosines gives $\beta_2=\text{acos}((1+d^2)/(2\sqrt{2}d))$. We arrive at $\theta_2=\pi/2-\alpha_2-\beta_2=\pi/2-\text{acos}((1-y)/\sqrt{(x^2+y^2+2x-2y+2)})-\text{acos}((x^2+y^2+2x-2y+3)/(2\sqrt{2}\sqrt{(x^2+y^2+2x-2y+2)}))$. A closed form solution is not possible because the above integrands lead to an elliptical integral. Relying on numerical integration methods leads to a probability of about 2.475% that there is no overlap when the center lies in R_1, and a probability of about 0.019% that there is no overlap when the center lies in R_2. The total probability is the sum of these two results, which with rounding gives about 2.49%.

Computer Bonus The longest string is **94 integers.** In the range from 1 to 1 billion, there is exactly one run of 94 integers from 981,270,903 to 981,270,996, inclusive, that contains no semiprimes.

NEW SPRING PROBLEMS

1 Solve the following cryptic addition in base 11. All the usual rules apply; each different letter stands for a different digit, and each different digit is always represented by the same letter; there are no leading zeros. Use *a* to represent the digit 10.

TICKLER = STRAINS + BRAINS

—**Howard G. McIlvried, III,** *PA Γ '53*

2 What is the maximum number of knights that can be placed on a standard 8×8 chess board, so that each knight threatens (i.e., can capture) exactly one other knight, and how are they arranged? A knight moves two squares in one direction and one square in another direction to end up on a square of opposite color. This move can occur even if intervening squares are occupied. In this puzzle, each knight can move to exactly one square occupied by another knight, but not to two or more occupied squares. Present your answer as an 8×8 grid (representing a chessboard) with the locations of the knights represented by Ns and the unoccupied squares represented by dashes.

—*A Passion for Mathematics* by Clifford A. Pickover

3 I am getting a new sign for my shop, and instead of showing the shop number as numeric digits, it will be spelled out in capital letters, one word per digit. For example, if my shop number were 103, it would be painted as ONE ZERO THREE. The sign painter is a little eccentric, and instead of charging by the hour, he charges by the brush stroke, which can be any shape and can touch, but must not cover, an already painted area (except where changing direction). He paints capital letters in a simple style (no serifs) and does not use two strokes where a single stroke will do. For example, E, F, G, H, I, N, O, R, S, T, U, V, W, X, Z would require 2, 2, 2, 3, 1, 1, 1, 1, 1, 2, 1, 1, 1, 2, and 1 strokes, respectively. My shop number, which I had spelled out for the painter when he arrived, is a prime number, consisting of three different digits. He charges $1 per stroke, regardless of the length of the stroke. By coincidence, the cost of my number in dollars was equal to the sum of the three digits in the shop number and was a prime. What is my shop number?

—Enigma in *New Scientist* by Susan Denham

4 Joe is playing five-card draw poker. He is dealt the 4, 5, and 6 of hearts, the 3 of diamonds, and the 10 of spades. He decides to discard the 3 and 10 and draw two new cards. What is the probability that he will improve his hand by getting one pair or better? Assume that Joe sees only the cards dealt to him and knows nothing about the other players' cards. In five-card draw, each player is dealt five cards face down. After looking at the cards, if he is not satisfied, he can turn in up to three cards (face down) and get replacements.

—Howard G. McIlvried, III, *PA Γ '53*

5 We are all familiar with the story of the 1001 Arabian Nights. Find N such that $N!$ (N factorial) has one digit for each of the Arabian Nights.

—H. S. Uhler

Bonus Find three consecutive triangular numbers whose product is a perfect square. The first such set is $6\times10\times15 = 900 = 30^2$, and the second such set is $300\times325\times351 = 34{,}222{,}500 = 5850^2$? What is the third such set, and what is the square root of their product? A triangular number is a number of the form $n(n + 1)/2$, where n is a positive integer.

—*Elementary Number Theory* by David M. Burton

Computer Bonus What is the largest base-10 prime, such that when the leading digit is successively removed until only one digit remains, the resulting series of integers, including the final digit, are all primes? Leading zeros are not allowed, and 1 is not a prime number.

—*Penguin Dictionary of Curious and Interesting Numbers* by David Wells

Postal mail your answers to any or all of the Brain Ticklers to **Tau Beta Pi, P. O. Box 2697, Knoxville, TN 37901-2697** or email to *BrainTicklers@tbp.org* as plain text only. The cutoff date for entries to the spring column is the appearance of the Summer *Bent* which typically arrives in mid-June (the digital distribution is several days earlier). The method of solution is not necessary. The Computer Bonus is not graded. We additionally welcome any interesting problems that might be suitable for the column. Entries will be forwarded to the judges who are **F.J. Tydeman,** *CA Δ '73*; **J.C. Rasbold,** *OH A '83;* **J.R. Stribling,** *CA A '92* and the columnist for this issue,

—H.G. McIlvried, III, *PA Γ '53*

"THE BEST PEOPLE" ENGINEERING JOB BOARD

Through a partnership with YourMembership, Tau Beta Pi is proud to offer a job board for students and alumni. Members can post resumes, browse over 5,400 engineering jobs, faculty positions, and internships, and employers may browse resumes.

New opportunities are posted on our home page daily and a full list of openings are available by visiting *www.tbp.org/memb/jobBoard.cfm.*

VOLUNTEERS WANTED

Every three years, the Executive Council is charged with appointing directors of various Association programs. The Council is seeking interested members to fill the positions of Director of Alumni Affairs, Director of the District Program, Director of Engineering Futures, Director of Fellowships, and Director of Rituals. Those selected to fill these roles must be willing and able to serve from July 1, 2019, until June 30, 2022.

Each position has unique qualifications, desired experience, and time commitment, which are outlined on the website at *www.tbp.org/about/programDirectors.cfm*. For all five positions, the Council is looking for volunteers who are passionate about Tau Beta Pi and interested in giving of their time and talent to benefit the Association and our members. Travel requirements vary for each role, but the ability to attend the annual Convention is required. Past experience and knowledge of the Association and its operations is required. The deadline to submit an application is May 15, 2019. It should be addressed to "The Executive Council" and emailed to *tbp@tbp.org*. There is no application form, but interested individuals should include a current resume and a letter stating the applicant's interest in the position, relevant past experience that would benefit the Association, and summary of ideas and vision for the position. If you would like additional information, please send an email to *tbp@tbp.org*. A member of the Executive Council will follow up with you about your interest or questions.