NEW TICKLERS

Fall Review
The most difficult problem in the Fall set was No. 3 about the average life span in Randomovia. In fact, we had more correct answers to the Fall Bonus than we did to No. 3.

New Problems
For your entertainment, here are the new Spring Ticklers.

1. A farmer has a hemispherical tub with a radius of 1 m, which he uses to water his cattle. What is the depth of the water when the tub is half full of water? Provide an exact answer in terms of accepted mathematical functions and symbols, such as sin, ln, e, etc., not a decimal approximation.

2. Let \( f(n) \) be the number of times the digit 1 is used in printing all the digits from 1 to \( n \) inclusive. Thus, for example, \( f(3) = 1 \), and \( f(12) = 5 \). Now, \( f(1) = 1 \). What is the next value of \( n \) for which \( f(n) = n \)?

3. The M. G. C. (Mathematicians’ Goofy Golf Club), to which I belong, always holds its annual banquet in July, and I was trying to find the date from some fellow members who were being annoyingly obtuse. One told me the date was an odd number, another that it was greater than 13, a third that it was a perfect cube. It is in base 10, and all the usual rules apply; that is, there are no leading zeros, and there is a one-to-one correspondence between digits and letters.

4. Solve the following cryptic multiplication (in which the \( X \) is a multiplication sign, not a letter in the cryptic). It is in base 10, and all the usual rules apply; that is, there are no leading zeros, and there is a one-to-one correspondence between digits and letters.

5. Solve the following equation:

\[
\frac{3}{4} = \frac{r}{2k} + \frac{1}{4} \]

Winter Answers
Now that we have provided you with some spring fun, here are the answers to the Winter Ticklers.

1. Because it is thin, it can be assumed that the tape on the takeup reel forms a perfect circle, with a radius of \( r \), whose prime divisors of its square equals a perfect cube. One and \( N \) are both divisors of \( N \).

2. Any string of seven identical digits is divisible by 1,111,111, whose prime factors are 239 and 4,649. Since 4,649 is...
does not have all different digits, it can’t be the combination. To generate the other integers with seven all-the-same digits, we have to multiply by 2, 3, 27, 5, 2 × 3, 7, 21, and 32. Use these factors times 4,649 and 239 to try to find two numbers such that both have all different digits and the larger contains all the digits in the smaller. For example, for 6,666,666, we have the possibilities of 239 × 27,894, 478 × 13,947, 717 × 9,298, and 1,454 × 4,649, but none of these satisfies our requirements. Trying all possibilities yields as the only solution the combination 37,192. The number George memorized is 37,192, and the number his wife produced is 8,888,888.

3 The cases of no birthdays in January, no birthdays in February, no birthdays in March, etc. are not mutually exclusive, since more than one of these events can occur at the same time. The probability that one or more of n not mutually exclusive events will occur is given by:

\[ P = \sum P(A_i) - \sum P(A_i \cap A_j) + \sum P(A_i \cap A_j \cap A_k) - \cdots \]

where \( P(A_i) \) is the probability that \( A_i \) occurs; \( P(A_i \cap A_j) \) is the probability that both \( A_i \) and \( A_j \) occur; etc. Since the events are not mutually exclusive, \( \Sigma P(A_i) \) overestimates the probability that one or more of \( A_i \) will occur, since both \( P(A_i) \) and \( P(A_i \cap A_j) \) include \( P(A_i) \). Therefore, \( \Sigma P(A_i \cap A_j) \) must be subtracted, but this overcorrects \( P_i \) and \( \Sigma P(A_i \cap A_j \cap A_k) \) must be added back. The same reasoning explains the other terms. For our problem, \( \Sigma P(A_i) = C(12,11)(11/12)^{27} \), \( \Sigma P(A_i \cap A_j) = C(12,10)(10/12)^{27} \) etc. Therefore, \( P = C(12,11)(11/12)^{27} - C(12,10)(10/12)^{27} + C(12,9)(9/12)^{27} - C(12,8)(8/12)^{27} + \cdots = 0.74122 - 0.19308 + 0.02210 - 0.00115 + \ldots = 0.56912 \). This problem can also be solved by creating a probability tree on a spreadsheet which spreads from the trunk at the top to 12 columns wide at the twelfth person’s row and then continues down to the 32nd person’s row. Far from being rare, more often than not for 32 randomly chosen people, there will be a month in which no one has a birthday.

4 Establish a coordinate system with \( y \) up the wall and \( x \) along the floor, and let 0 be the angle the ladder makes with the floor. The equation for the ladder is then given by \( y = L \sin \theta - x \tan \theta \), where \( L \) is the length of the ladder. As the ladder slides down the wall, the maximum value of \( y \) at a given value of \( x \) will be a point on the boundary curve. We can find this value by setting the derivative of \( y \) with respect to \( x \) equal to 0 while keeping \( x \) constant. That is, \( dy/d\theta = 0 \). Therefore, \( x = L \cot \theta \). Substituting this back in the equation for \( y \) gives \( y = L \sin \theta \), but since \( \sin \theta + \cos \theta = 1 \), we have \( x(L)^{2n} + (y(L)^{2n} = 1, or \( x^{2n} + y^{2n} = L^{2n} \), which is the equation of the boundary curve.

5 Let \( r_1 \) and \( r_2 \) be the top and bottom radii of the candle in cm. Then, the radius at any distance from the top is given by \( r = r_1 + ay \), where \( a = (r_1 - r_2)/45 \), and \( y \) is the distance from the top in cm. The equation for the volume of the frustum of a right circular cone is \( V = \pi h (r_1^2 + r_1 r_2 + r_2^2) \), where \( r_1 \) and \( R \) are the radii of the top and bottom and \( h \) is the altitude. Thus, we have \( V_i = 15\pi r_i^2 + r_i (r_i + 45\alpha) + (r_i + 45\alpha)^2 \) \( = 15\pi (r_i^2 + 135\alpha + 2.025\alpha^2) \), \( V_i = \pi (r_i^2 + r_i + 3a) + (r_i + 3a)^2 \) \( = \pi (3r_i^2 + 9ar_i + 9a^2) \), and \( V_i = \pi (r_i + 42a)^2 + (r_i + 42a)(r_i + 45a) + (r_i + 45a)^2 \) \( = \pi (3r_i^2 + 261r_i + 567a^2) \). Since burning time is proportional to volume, \( t_i = (V_i - V_j)/V_i \). Using \( t_i - t_j = 20 \) min., \( t_j = 540 \) min., and the values for \( V_i, V_j, \) and \( V_i \) and making the substitution \( z = r_i/a \), we get \( 3z^2 + 30z - 13,635 = 0 \), which can be solved by the quadratic formula to get \( z = 127.576 \), which allows the calculation of \( V_j/a^2 \) and \( V_j/a^2 \). Then, from the relationship \( t_i = (V_i/V_j) \) we can calculate \( t_i = 26.433 \) min, the time for the top 3 cm to burn.

Bonus. Before Pete says anything, Sam knows that Pete does not know the numbers. Therefore, the sum must be a number that cannot be the sum of two primes, nor the cube of a prime, nor a prime times a composite number, where the products of the prime times all factors of the composite are greater than 100, nor a number near 200, where only one possibility exists, such as 198, which must be 99 + 99. A few quick calculations show that the sum of \( A \) and \( B \) must be 11, 17, 23, 27, 29, 35, 37, 41, 47, 51, or 53. Evaluating the products for all possible \( A \) and \( B \) combinations for the 11 possible sums shows that 10 of them share a possible product with another potential sum. Only 17 has a unique product, 52. Thus, Pete’s number is 52, and Sam’s number is 17, which leads to \( A \) and \( B \) having values of 4 and 13.

Double Bonus: Although this problem does not have a unique solution, one quite good “nine-digit” approximation to \( \pi \) may be arrived at by the following method. From \( 1/(\pi - 3) = 7.062513 \), we see that \( 3 + 1/(7 + 4^2) \) is a good start. Next, calculate \( (\pi - 3)/(17.0625) = 0.999998116 \), which means that the fractional part of our approximation is too large. Subtracting the ratio from 1 gives 0.000001884. Trying various possibilities with the remaining digits of 5, 6, 8, and 9, we find that \( 9^6 = 0.000001882 \), slightly smaller than desired, so we use the two remaining digits to modify this slightly, to give \( \pi \equiv 3 + \frac{1 - 9^6}{2} = 3.14159265390 \) accurate to nine decimal places, compared to the eight decimal-place accuracy of the example in the Winter column.