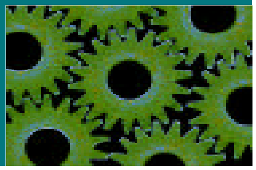

NEW TICKLERS

The judges are: F.J. Tydeman, *CA* Δ *'73;* R.W. Rowland, *MD* Β *'51;* D.A. Dechman, *TX* Α *'58;* and the columnist for this issue,

Howard G. McIlvried III, *PA* Γ *'53.*

Fall Review

The most difficult problem in the Fall set was No. 3 about the average life span in Randomovia. In fact, we had more correct answers to the Fall Bonus than we did to No. 3.

New Problems

For your entertainment, here are the new Spring Ticklers.

1 A farmer has a hemispherical tub with a radius of 1 m, which he uses to water his cattle. What is the depth of the water when the tub is half-full of water? Provide an exact answer in terms of accepted mathematical functions and symbols, such as sin, ln, e, π, etc., not a decimal approximation.

— **C.K. Galer,** *MI* Α *'77*

2 Let $f(n)$ be the number of times the digit 1 is used in printing all the digits from 1 to n inclusive. Thus, for example, $f(3) = 1$, and $f(12) = 5$. Now, $f(1) = 1$. What is the next value of n for which $f(n) = n$?

— *Technology Review*

3 The M. G. G. C. (Mathematicians' Goofy Golf Club), to which I belong, always holds its annual banquet in July, and I was trying to find the date from some fellow members who were being annoyingly obtuse. One told me the date was an odd number, another that it was greater than 13, a third that it was not a perfect square, and a fourth that it was a perfect cube. Finally, W. W. Webb, president of the club, told me that the date was less than my highest single hole score last season (which was, in fact, 17, and I thought it was a little tacky of Webb to bring that up). Later, when I finally learned the date, I discovered that only one of the five statements was correct. What was the date of the banquet?

— *Adapted from* Brain Puzzlers Delight *by E.R. Emmet*

4 Solve the following cryptic multiplication (in which the **×** is a multiplication sign, not a letter in the cryptic). It is in base 10, and all the usual rules apply; that is, there are no leading zeros, and there is a one-to-one correspondence between digits and letters.

SINK × THEM = DEEPDEEP

— *Technology Review*

5 An urn contains 80 balls, 72 green and 8 red. The balls are drawn randomly one at a time without replacement until all the red balls are drawn. What is the expected value of the number of balls remaining in the urn at that point?

— William Allen Whitworth (1901)

Bonus. A boy is standing on a dock looking at the water in a smooth pond, when the ball he is holding slips from his hand. As he watches it disappear below the surface, he fears that it is gone forever. However, he is relieved a short time later to see it reappear. The ball, which has a diameter of 6 cm and a mass of 112 gm, falls from rest at a height of 2 m from the center of the ball to the surface of the water. Although not completely accurate, for the purposes of this problem assume that the drag force on the ball in the water equals kAv^2 where v is the ball's velocity in m/s, A is the cross sectional area of the ball in m², and k has the value of 353.7 Newton s²/m⁴. Ignore air resistance, and assume that the ball encounters no drag or buoyancy forces in the water until the ball is half submerged, at which point these forces come into full play. Use a density of 1,000 kg/m³ for the water and a value of 9.8 m/sec² for the acceleration due to gravity. To three significant figures, how many seconds elapse between the time the ball leaves the boy's hand until it just breaks the surface upon reappearing?

— **John W. Langhaar,** *PA* Α *'33*

Computer Bonus. Find the smallest integer greater than one which has the property that the sum of all the integral divisors of its square equals a perfect cube. One and N are both divisors of N.

— *Pierre de Fermat (circa 1650)*

Winter Answers

Now that we have provided you with some spring fun, here are the answers to the Winter Ticklers.

1 Because it is thin, it can be assumed that the tape on the takeup reel forms a perfect circle, with a radius of $r = r_0 + Tkc$, where r_0 is the initial radius of the takeup reel, T is the thickness of the tape, c is the counts, and k is the proportionality factor between counts and revolutions. The length of the tape on the reel is $L = st$, where s is the speed of the tape, and t is the time. Then, the cross sectional area of the tape is given by $A = LT = sTt = \pi(r^2 - r_0^2)$. Substituting for r and simplifying gives $t = (\pi/sT)(T^2k2c^2 + 2\,r_0Tkc) = ac^2 + bc$, where $a = \pi Tk^2/s$ and $b = 2\pi r_0k/s$. Differentiating gives $dt/dc = 2ac + b$. When $c = 1{,}000$, $dc/dt = 1$, which gives $2{,}000a + b = 1$. A second equation is developed by solving for c in terms of t using the quadratic equation and substituting in the equation for dt/dc to get $dt/dc = \sqrt{(b^2 + 4at)}$. When $t = 2{,}000$, $dc/dt = 2/3$. Therefore, $3/2 = \sqrt{(b^2 + 8{,}000a)}$. Solving these equations simultaneously gives:

$a = 1/4{,}000$ and $b = 1/2$.

Therefore, $t = c^2/4{,}000 + c/2$ seconds.

2 Any string of seven identical digits is divisible by 1,111,111, whose prime factors are 239 and 4,649. Since 4,649 does not have all different digits, it can't be the combination. To generate the other integers with seven all-the-same digits, we have to multiply by 2, 3, 2^2, 5, 2 × 3, 7, 2^3, and 3^2. Use these factors times 4,649 and 239 to try to find two numbers such that both have all different digits and the larger contains all the digits in the smaller. For example, for 6,666,666, we have the possibilities of 239 × 27,894, 478 × 13,947, 717 × 9,298, and 1,434 × 4,649, but none of these satisfies our requirements. Trying all possibilities yields as the only solution the combination 37,192. The number George memorized is 239, and the number his wife produced is 8,888,888.

3 The cases of no birthdays in January, no birthdays in February, no birthdays in March, etc. are not mutually exclusive, since more than one of these events can occur at the same time. The probability that one or more of n not mutually exclusive events will occur is given by:

$P = \Sigma P(A_i) - \Sigma P(A_i A_j) + \Sigma P(A_i A_j A_k) - ... + (-1)^{n-1} \Sigma P(A_i A_j A_k ... A_n)$

where $P(A_i)$ is the probability that A_i occurs; $P(A_i A_j)$ is the probability that both A_i and A_j occur; etc. Since the events are not mutually exclusive, $\Sigma P(A_i)$ overestimates the probability that one or more of A_i will occur, since both $P(A_i)$ and $P(A_j)$ include $P(A_i A_j)$. Therefore, $\Sigma P(A_i A_j)$ must be subtracted, but this overcorrects P, and $\Sigma P(A_i A_j A_k)$ must be added back. The same reasoning explains the other terms. For our problem, $\Sigma P(A_i) = C(12,11)(11/12)^{32}$, $\Sigma P(A_i A_j) = C(12,10)(10/12)^{32}$, etc. Therefore, $P = C(12,11)(11/12)^{32} - C(12,10)(10/12)^{32} + C(12,9)(9/12)^{32} - C(12,8)(8/12)^{32} + ... = 0.74122 - 0.19308 + 0.02210 - 0.00115 + ... = 0.56912$. This problem can also be solved by creating a probability tree on a spreadsheet which spreads from the trunk at the top to 12 columns wide at the twelfth person's row and then continues down to the 32nd person's row. Far from being rare, more often than not for 32 randomly chosen people, there will be a month in which no one has a birthday.

4 Establish a coordinate system with y up the wall and x along the floor, and let θ be the angle the ladder makes with the floor. The equation for the ladder is then given by $y = L\sin\theta - x\tan\theta$, where L is the length of the ladder. As the ladder slides down the wall, the maximum value of y at a given value of x will be a point on the boundary curve. We can find this value by setting the derivative of y with respect to θ equal to 0 while keeping x constant. That is, $dy/d\theta = 0 = L\cos\theta - x\sec^2\theta$. Therefore, $x = L\cos^3\theta$. Substituting this back in the equation for y gives $y = L\sin^3\theta$, but since $\sin^2\theta + \cos^2\theta = 1$, we have $(x/L)^{2/3} + (y/L)^{2/3} = 1$, or $x^{2/3} + y^{2/3} = L^{2/3}$, which is the equation of the boundary curve.

5 Let r_t and r_b be the top and bottom radii of the candle in cm. Then, the radius at any distance from the top is given by $r = r_t + ay$, where $a = (r_b - r_t)/45$, and y is the distance from the top in cm. The equation for the volume of the frustum of a right circular cone is $V = (\pi h/3)(r^2 + rR + R^2)$, where r and R are the radii of the top and bottom and h is the altitude. Thus, we have

$V_c = 15\pi[r_t^2 + r_t(r_t + 45a) + (r_t + 45a)^2]$
$= 15\pi(3r_t^2 + 135ar_t + 2{,}025a^2)$,
$V_t = \pi[r_t^2 + r_t(r_t + 3a) + (r_t + 3a)^2]$
$= \pi(3r_t^2 + 9ar_t + 9a^2)$, and
$V_b = \pi[(r_t + 42a)^2 + (r_t + 42a)(r_t + 45a) + (r_t + 45a)^2]$
$= \pi(3r_t^2 + 261ar_t + 5{,}679a^2)$.

Since burning time is proportional to volume, $(t_b - t_t)/t_c = (V_b - V_t)/V_c$. Using $t_b - t_t = 20$ min., $t_c = 540$ min., and the values for V_c, V_b, and V_t and making the substitution $z = r_t/a$, we get $5z^2 - 531z - 13{,}635 = 0$, which can be solved by the quadratic formula to give $z = 127.576$, which allows the calculation of V_t/a^2 and V_c/a^2. Then, from the relationship $t_t = t_c(V_t/V_c)$ we can calculate $t_t = 26.433$ min, the time for the top 3 cm to burn.

Bonus. Before Pete says anything, Sam knows that Pete does not know the numbers. Therefore, the sum must be a number that cannot be the sum of two primes, nor the cube of a prime, nor a prime times a composite number, where the products of the prime times all factors of the composite are greater than 100, nor a number near 200, where only one possibility exists, such as 198, which must be 99 + 99. A few quick calculations show that the sum of A and B must be 11, 17, 23, 27, 29, 35, 37, 41, 47, 51, or 53. Evaluating the products for all possible A and B combinations for the 11 possible sums shows that 10 of them share a possible product with another potential sum. Only 17 has a unique product, 52. Thus, Pete's number is 52, and Sam's number is 17, which leads to A and B having values of 4 and 13.

Double Bonus: Although this problem does not have a unique solution, one quite good "nine-digit" approximation to π may be arrived at by the following procedure. From $1/(\pi - 3) = 7.062513$, we see that $3 + 1/(7 + 4^{-2})$ is a good start. Next, calculate $(\pi - 3)/(1/7.0625) = 0.999998116$, which means that the fractional part of our approximation is too large. Subtracting the ratio from 1 gives 0.000001884. Trying various possibilities with the remaining digits of 5, 6, 8, and 9, we find that $9^{-6} = 0.000001882$, slightly smaller than desired, so we use the two remaining digits to modify this slightly, to give

$$\pi \quad 3 + \frac{1 - 9^{-6+8^{-5}}}{7 + 2^{-4}} \quad 3.14159265390$$

accurate to nine decimal places, compared to the eight decimal-place accuracy of the example in the Winter column.