Brain Ticklers

RESULTS FROM SPRING

Perfect			
Beckham III, C. Lee	IN	Β	'86
Bohdan, Timothy E.	IN	Γ	'85
Couillard, J. Gregory	IL	Α	'89
*Gerken, Gary M.	CA	Η	'11
*Griggs Jr., James L.	OH	Α	'56
*Gulian, Franklin J.	DE	Α	'83
Gulian, Joseph D.	Son of member		
Heske III, Theodore	PA	Α	'86
*Norris, Thomas G.	OK	Α	'56
Norris Jr., Thomas G.	PA	Γ	'79
*Schmidt, V. Hugo	WA	Β	'51
*Slegel, Timothy J.	PA	Α	'80
*Strong, Michael D.	PA	Α	'84
Other			
Alexander, Jay A.	IL	Γ	'86
Aron, Gert	IA	Β	'58
Celani, Paul E.	MD	Γ	'97
Christiansen, Reed L.	MN	Α	'83
Dechman, Don A.	TX	Α	'57
Ellis Jr., Ira T.	PA	Η	'56
Janssen, James R.	CA	Γ	'82
Johnson, Roger W.	MN	Α	'79
Jones, John F.	WI	Α	'59
Jones, Jeffrey C.	Son of member		
Jordan, R. Jeffrey	OK	Γ	'00
Krischer, Ari Y.	NY	Δ	'19
Lalinsky, Mark A.	MI	Γ	'77
McHenry, S. Dale	MO	Β	'81
Parks, Christopher J.	NY	Γ	'82
Penlesky, Richard J.	WI	Β	'73
Richards, John R.	NJ	Β	'76
Riedesel, Jeremy M.	OH	Β	'96
Rigge, Paul J.	MI	Γ	'12
Sigillito, Vincent G.	MD	Β	'58
*Spong, Robert N.	UT	Α	'58
Summerfield, Steven L.	MO	G	'85
Voellinger, Edward J.	Non-member		
*Wells, Alan T.	RI	Β	'16

*Denotes correct bonus solution

SPRING REVIEW

The Spring regular problems appear to have been a relatively easy set with an average of 85% correct answers. The easiest regular problem was No.1 about the 5 nine-digit numbers; and the most difficult was No. 2 about the probability of a particular bridge hand. Regarding the Spring Bonus, the judges goofed and published an incorrect answer based on the assumption that the center of gravity (COG) of the cone was coincident with the center of mass (COM). However, this is true only if the gravity field is uniform. This approach gives g_a=2.75 m/s^2 and g_b=24.8 m/s^2. The correct approach is to set up an equation giving the gravitational force between a particle of mass of the cone and its apex or base point and calculating the total gravitational force by triple integration, similar to the method presented for calculating the COM of the cone in the incorrect solution in the Summer column. This approach gives g_a=3.92 m/s^2 and g_b=9.49 m/s^2. The judges apologize for any inconvenience this may have caused.

SUMMER ANSWERS

1 They arrive at **1:00 pm.** The two family members each walk exactly the same distance as the other rides. Let w_d and w_f be the distances that the daughter and father each walk the first half of the trip. Then $w_d / 8 + w_f / 16 = w_d / 16 + w_f / 6$. Since $w_f = 32 - w_d$, rewrite the equation as $w_d / 8 + (32 - w_d) / 16 = w_d / 16 + (32 - w_d) / 6$. Simplifying gives $w_d = 20$ km and $w_f = 12$ km. In the first half, the daughter walks for 2½ hours and rides ¾ hours, the father walks 2 hours and rides 1¼ hours. Each half takes 3¼ hours plus ½ hour rest for a total trip of 7 hours.

2 The smallest number has **99** consecutive 6's. Let S(n) be the integer made up of n consecutive 6's in base ten notation. S(0) = 0, S(1) = 6_{10} = 6_{16}, S(2) = 66_{10} = 42_{16}, S(3) = 666_{10} = $29A_{16}$ and so forth. S(n) = $6\,\Sigma_{i=0}^{\,i-1}(10^i)$ = $2(3)\,\Sigma_{i=0}^{\,i-1}(2^i5^i)$. Recursively, S($n$) = S($n$–1) + $6(10^{n-1})$ = S(n–1) + $3(2^n)(5^{n-1})$. The factor 2^n in the second addend ensures that the lower n bits of S(n) are identical to those in S(n–1). Each additional 6 prepended to the base 10 representation of S(n) makes invariant one additional least significant bit; it follows that every four 6's makes a hexit invariant. S(3) ends with one A, S(7) ends with two A's, and by induction S(99) ends with 25 A's.

3 The two numbers are **5** and **14.** This simple algebra problem can be represented by two equations. First, $2l = 5s + 3$ or $4l = 10s + 6$. Second, $4l + 3s = 71$, which can be rewritten as $4l = -3s + 71$. Combining the two gives $13s = 65$, giving $s = 5$ and $l = 14$.

4 There are **201** people, and **19999** turnips in the original pile. Let T be the number of turnips. Call N the number of people, and since the quotient T/N is between 99 and 100 but closer to 99, we can write $99 < T/N < 99.5$ or $198N < 2T < 199N$. The second constraint that the number of turnips divided by 100 is closer to N than the number of turnips divided by 99, that is, $(N - T/100) < (T/99 - N)$, or $19800N < 197T$, provides a tighter lower bound than the first constraint. Combining the two, we get the bounds $39600N < 398T < 39601N$. N must be odd, because if N is even, there's no T that strictly meets the less-than upper bound. And for odd $N <= 199$, no integer T fits both bounds. $N = 201$ gives $T = 19999$. The next higher $N = 203$ gives $T = 20197$ which is too large.

5 **Friday** is the day least likely to form a pair from the first three socks chosen. However, **Saturday** is the day which requires the greatest number of draws of the three socks to make a pair.

On a given day, a drawer is characterized by the distribution of the number of socks of each color it contains. Consider a drawer with n_0, n_1, and n_2 socks. Let $n_{tot} = n_0+n_1+n_2$. Three socks can be randomly drawn in C(n_{tot}, 3) ways, where C stands for the Choose function. A selection by the uncle can pick exactly one sock of each color in $n_0n_1n_2$ ways. Therefore, the probability of not drawing a pair is $q = n_0n_1n_2$ / C(n_{tot}, 3).

Let p be the probability of getting a match on the first draw, that is, $p = 1-q$. The expected number of selections is

$E = p(1 + 2q + 3q^2 + 4q^3 + \ldots)$ which simplifies to $E=1/p$ for $0 < p <= 1$.

On Monday, prior to a draw, there are 16 socks and the drawer has only one possible configuration $(n_0, n_1, n_2) = (8,6,2)$. For this configuration, $q=6/35$, $p=29/35 \cong 0.8286$, and $E=35/29 \cong 1.2069$.

On Tuesday, there are 14 socks and they can be distributed in three different ways: (6,6,2), (8,4,2) and (8,6,0). The likelihood of each possible Tuesday distribution can be calculated from Monday's distribution. That is, the probability of drawing a match when there are n_i socks on Monday is $(C(n_i, 3) + C(n_i, 2)C(n_{tot}-n_i, 1)) / (C(n_{tot}, 3) - n_0n_1n_2)$.

The first term in the numerator is the ways all three socks can match and the second term is the ways two socks can match times the ways the third sock can be picked. The first term in the denominator is the total ways that three socks can be drawn; the second term is the ways three socks can be picked with no match, so the equation gives the probability of getting a pair of socks of a given color.

Specifically, on Tuesday, the (6,6,2), (8,4,2), and (8,6,0) distributions occur with frequencies 35/58, 85/232, and 7/232, respectively. Calculate p and E for each distribution, scale by their rate of occurrence, and for Tuesday, the probability of match on the first draw is $2154/2639 \cong 0.8162$ and the expected number of draws is $38962/31755 \cong 1.2270$.

Continue for Wednesday, Thursday, Friday, and Saturday. Calculations are made less tedious if one observes that because the uncle only cares about making a match, but not the color of the matching socks, two distributions can be considered the same if the sock counts match independent of color. Also, observe that all configurations with a zero n_i can be treated identically, having $p = 1$ and $E = 1$.

The following table illustrates match on first draw probabilities and expected number of draws for each day, to four decimal places:

Day	Match	Expected
Mon	0.8286	1.2069
Tue	0.8162	1.2270
Wed	0.8027	1.2511
Thu	0.7892	1.2801
Fri	0.7805	1.3074
Sat	0.7849	1.3586

Surprisingly, the chances of matching dips on Friday below the other days, while Saturday predicts the highest number of expected draws. The calculations are clearly easier with a program, spreadsheet, or calculator, but are possible using rational fractions. The exact Friday match probability is 698897/895491, and the actual Saturday expected value is 36497826/26864730.

Bonus The judges found that the blue die can beat white in as little as ⅓ of the time. One possible number distribution is **blue** has **6, 7, 8, 9, 23, 24; green** has **3, 4, 5, 20, 21, 22; red** has **1, 2, 16, 17, 18, 19; white** has **10, 11, 12, 13, 14, 15.** Note the symmetry of the solution. Each die beats the next one ⅔ of the time, so there are three other solutions, found by rotating the sets of six numbers through the four dice.

Suppose a fifth, amber colored die was introduced that beat blue ⅔ of the time. We could number the thirty faces⅓ 1 through 30 as follows: amber 10, 11, 12, 13, 14, 15; blue 6, 7, 8, 9, 29, 30; green 3, 4, 5, 26, 27, 28; red 1, 2, 22, 23, 24, 25; white 16, 17, 18, 19, 20, 21. Curiously, each die would beat the succeeding one ⅔ of the time, but amber never beats white.

Double Bonus A two spot game can end in **19** different ways. In two spot Sprouts, there are initially six lives, three for each of the original spots. Each player's turn decreases the life of each connected spot by one, but adds one at the newly drawn spot. Two-spot Sprouts determines a winner in four or five moves. There is one initial configuration, two after one move, six after two, 11 after three, 17 after four (of which five are final), and 14 after five (all final). A deeper analysis of Sprouts, as well as a diagram of the complete two spot game tree can be found in *Computer Analysis of Sprouts* by Applegate, Jacobson, and Sleator.

NEW FALL PROBLEMS

1 Solve this cryptic addition with the usual rules: different letters are different digits, same letter is same digit, no leading zeros, base 10: SIXTEEN + TWENTY + TWENTY + TEN + TWO + TWO = SEVENTY.

—*Journal of Recreational Mathematics*

2 Hook, Line, and Sinker returned from a day's fishing, and each reported his catch secretly, but accurately, to George Gaff, landlord of The Complete Idiot. "Well, Gents," George announced later. "Hook caught the most fish, and Sinker caught the fewest (no ties). If you divide Hook's catch by Sinker's catch, you get Line's catch."

Upon reflection, Hook remarked, "I know how many fish each of us caught."

Line then chimed in with, "I also know how many each of us caught."

"But I don't know," Sinker complained after a short pause.

"Never mind, Old Chap," said George. "I'll give you a clue. I've been fishing, too, and I caught fewer fish than Hook. If you know how many I caught, you can figure out how many fish Line caught." Without further help, Sinker managed to work out Line's catch. Sinker was proud of himself and remarked that it had been a good day, since his catch was twice what he had caught the previous week. How many fish did Hook, Line, and Sinker each catch?

—A Tantalizer by Martin Hollis in *New Scientist*

3 In how many different ways (order does not matter) can one change a $100 bill using $1, $2, $5,

$10, $20, and $50 bills?

—*Challenging Mathematical Problems with Elementary Solutions* by A.M. & I.M. Yaglom

4 What is the maximum number of queens that can be placed on an 8x8 chessboard so that each queen threatens (that is, could capture on the next move) exactly two other queens? A queen can move horizontally, vertically, or diagonally as many squares as desired or until another piece or the edge of the board is encountered. Using Q's and -'s, provide an 8x8 grid that represents the chess board.

—*The Colossal Book of Mathematics* by Martin Gardner

5 The There'sAPillForEverything Pharmaceutical Company received a shipment of 12 bottles of pills, each bottle containing 1,000 pills. The labels on the bottles indicated that each pill weighed 100 mg. However, before any pills had been sold, the pharmacist received a notice from the supplier that the manufacturer had some problems, and some of the pills might weigh 110 mg. However, if a 110 mg pill was found in a bottle, all the pills in that bottle would weigh 110 mg. The pharmacist has an ordinary pan scale that is accurate to 1 mg. What is the minimum number of weighings of pills from the various bottles required to determine which bottles, if any, contain pills weighing 110 mg? Explain how the weighings are to be carried out. The pan on the scale can hold at most 24 pills at the same time. Pills on the scale can be identified as to which bottle they came from.

—*The Crucible*

Bonus Two positive integers with no common integer factor (other than one) are selected. Their sum is written on the forehead of logician A and the sum of their squares is written on the forehead of logician B. The two logicians are allowed to see the numbers on each-other's foreheads. The following conversation takes place.

A (sees sum of squares): I don't know my number.
B (sees sum): I don't know my number.
A: I don't know my number.
B: I don't know my number.
A: I don't know my number.
B: I don't know my number.
A: I don't know my number.
B: I don't know my number.
A: I know my number.
What are the numbers on A's and B's foreheads?

—**Richard I. Hess,** *CA B '62*

Computer Bonus Find five different positive integers such that the sum of any two of them is a perfect square.

—Allan Gottlieb's Puzzle Corner in *Technology Review*

Postal mail your answers to any or all of the Brain Ticklers to **Tau Beta Pi, P. O. Box 2697, Knoxville, TN 37901-2697** or email to *BrainTicklers@tbp.org* as plain text only. The cutoff date for entries to the Fall column is the appearance of the Winter *Bent* which typically arrives in late December (the digital distribution is several days earlier). The method of solution is not necessary. We welcome any interesting problems that might be suitable for the column. The Computer Bonus is not graded. Entries will be forwarded to the judges who are **H.G. McIlvried III,** *P A Γ '53*; **J.C. Rasbold,** *OH A '83*, **J.R. Stribling,** *CA A '92*; and the columnist for this issue,

—**F.J. Tydeman,** *CA Δ '73*