Brain Ticklers

SUMMER SOLUTIONS

Readers’ entries for the Summer problems will be acknowledged in the Winter ’06 BENT. Meanwhile, here are the answers:

1. In this problem, we asked for not only the results of the round robin chess tournament, but also the occupations of the players. The tournament results are as follows (2=W, 1=Tie, 0=Loss):

   \[
   \begin{array}{cccccc}
   \text{Player} & \text{A} & \text{B} & \text{C} & \text{D} & \text{E} & \text{Score} \\
   \hline
   \text{A} & 1 & 2 & 2 & 0 & 5 \\
   \text{B} & 0 & 1 & 0 & 1 & 4 \\
   \text{C} & 0 & 0 & 1 & 0 & 1 \\
   \text{D} & 0 & 0 & 0 & 1 & 1 \\
   \text{E} & 0 & 0 & 0 & 0 & 0 \\
   \hline
   \end{array}
   \]

2. You were asked to find the maximum size of a set of \( N \) distinct positive integers, the sum of any \( K \) of which is prime, for the cases when \( K = 2, 3, 4, \) and 5. The first case to try is \( N = K; \) in this case (1, 2, 3, ..., \( K \)-1, X) can be shown to be a solution set, where 1 through \( K \) are consecutive integers and \( X \), greater than \( K \)-1, causes the sum of the set to be prime. When \( N \) is greater than \( K \), it is useful to compute the sums \( \text{mod}\) \( K \). When any sum of \( K \) integers is zero \( \text{mod}\) \( K \), then they are a multiple of \( K \) and hence not prime. By looking at the sums \( \text{mod}\) \( K \), you can show that (2, 3, ..., \( K \)-1, X) is a solution set. Then one can construct solution sets for (K, N) = (2, 2), (3, 4), and (4, 4). The \( K = 5 \) case is much harder. We used a computer to find a solution for (5, 7). The situation for (K, N) = (5, 8) is still open; we have neither a proof that it cannot be done nor a valid solution set. We intend to be lenient in grading. The following list presents the largest sets that we found for each case.

   \[
   \begin{array}{cccc}
   \text{K} & \text{N} & \text{example} \\
   \hline
   2 & 2 & (1,2) \\
   3 & 4 & (1,3,7,9) \\
   4 & 4 & (1,2,3,5) \\
   5 & 7 & (1,3,13,43,253,751,979) \\
   \end{array}
   \]

3. The smallest anti-magic square (all magic sums different) consisting of nine positive, not necessarily all different, integers, that we found was:

   \[
   1 1 1 \\
   1 2 4 \\
   3 3 5
   \]

As judged by the “smallness” criteria, in decreasing order of priority, the largest integer used is 5, the sum of the integers used is 19, and the sum of the integers in the first row is 3.

4. The problem asks for the rational values that may be assumed by the expression \( R = \sqrt{(N/7)^2 + 5,936} \), for \( N \) a positive integer. \( R \) can assume only three values: 61,692/7; 1,260; and 8,806/7 when \( N \) is 61,690; 8,806; and 1,210, respectively. If you express \( R \) as \( P/Q \), where \( P \) and \( Q \) are positive integers, then the expression for \( R \) can be restated as \( TP = Qv(N^2 + 246,764) \). This means that if \( N^2 + 246,764 \) is a perfect square, then the equation in \( P \) and \( Q \) will be solved for \( Q = 7 \) and \( P^2 = N^2 + 246,764 \). This later expression can be rewritten as \( (P + N)(P - N) = 246,764 \). Note that since \( 246,764 \) is even, both \( P + N \) and \( P - N \) must also be even. So, find all of the divisors \( (D_1) \) of \( 246,764 \) that leave an even quotient \( (D_2) \). The only three divisor pairs \( (D_1, D_2) \) are \( (2, 123,382) \), \( (14, 17,626) \), and \( (98, 2,518) \). Finally note that \( N = (D_1 - D_2)/2 \) and \( P = (D_1 + D_2)/2 \). This yields \( N = 61,690; 8,806; \) and 1,210, and \( R \) as stated above.

5. This problem asked for which of two different income averaging schemes, bottom-up or top-down, would the richest and the poorest income brackets in Puevigi prefer. Number the income groups from one to ten. When neighboring income groups are averaged in order from poorest to richest, the resulting nine income groups have values of \( (1.5, \ 2.25, 3.125, \ ..., 9.00195) \). When the averaging proceeds from richest to poorest, the resulting nine income groups have wealth values of \( (9.5, \ 8.75, 7.875, \ ..., 1.98805) \). Thus both the richest and the poorest income
The following intuitive solution is attributed to John Edson Sweet (You are a Mathematician by D. Wells). Imagine having three different spheres touching one another and sitting on a plane, say plane #1. Then there exists a second plane, plane #2, tangent to the three spheres but covering them from above. Planes #1 and #2 are bound to have a common line l, the line of their intersection. Finally, note that the bisector plane of the solid angle formed by planes #1 and #2 passes through the centers of all three spheres. This is the plane, plane #3, that contains the three externally tangent circles and the six external tangents. By definition, plane #3 passes through line l, and line 1 contains the three points of intersection.

**DOUBLE BONUS.** In this problem, you were asked to prove that the three points of intersection of the three pairs of lines that are the external tangents of three circles, each circle externally tangent to two others, lie on a straight line. Let A be the center of the largest circle (of radius R1), B the center of the middle circle (of radius R2), and C the center of the smallest circle (of radius R3). Let P be the intersection point of the tangents to circles A and B, Q be the intersection point of the tangents to circles A and C, and R be the intersection point of the tangents to circles B and C. Construct a fourth circle with center D such that its radius is the same as the radius of the smallest circle (R3) and it is tangent to both the external tangents of circles A and B. Then:

PD/PA = R/R3 = QC/QA

Hence, DC is parallel to PQ. In the same way, DC is parallel to segment PQ if part of line PR.

The following intuitive solution is attributed to John Edson Sweet (You are a Mathematician by D. Wells). Imagine having three different spheres touching one another and sitting on a plane, say plane #1. Then there exists a second plane, plane #2, tangent to the three spheres but covering them from above. Planes #1 and #2 are bound to have a common line l, the line of their intersection. Finally, note that the bisector plane of the solid angle formed by planes #1 and #2 passes through the centers of all three spheres. This is the plane, plane #3, that contains the three externally tangent circles and the six external tangents. By definition, plane #3 passes through line l, and line 1 contains the three points of intersection.

**NEW FALL PROBLEMS**

1. Solve the following cryptic addition:

   \[ \frac{A}{DE} + \frac{B}{FG} + \frac{C}{HJ} = 1 \]

   where each letter represents a different digit and none of them is zero. DE, FG, and HJ are each two-digit integers and none of them is zero. DE, FG, and HJ are each two-digit integers, e.g., DE = 16 * D + E. We want the solution in which A < B < C.

2. Rex lies a lot. In fact, he tells the truth on only one day of the week. One day, he said, “I lie on Mondays and Tuesdays.” The next day, he said, “Today is either Thursday, Saturday, or Sunday.” The next day, he said, “I lie on Wednesdays and Fridays.” On which day of the week does Rex tell the truth? (Mind Puzzlers by George J. Summers)

3. What is the dihedral angle between two faces of a regular tetrahedron? Express your answer exactly using integers and trigonometric and algebraic functions. —Paul J. Blatz

4. Here is the situation on a game show. As three contestants enter the stage, a red or blue hat is placed on each one’s head, depending on the flip of a fair coin. Each contestant can see the other’s hats but not his own, and they cannot communicate with each other in any way. They are told that if one or more of them can identify the color of hat he or she is wearing and no one gives a wrong answer, each will win a prize, but if no one answers or if one or more give a wrong answer, they will get nothing. However, before the hats are put on them, they are allowed to consult with each other. If they adopt the optimal strategy, what is their probability of winning?

   —Dr. Todd D. Ebert, PA B ’91
   University of California-Irvine

5. Assume that you have a frictionless hemisphere of radius R with its flat side fastened to a perfectly level table. You place a very small object on top of the hemisphere. At what height above the table does the object leave the hemisphere, and how far from the center of the hemisphere does it land?

   —Technology Review

6. Solve the following cryptic addition:

   \[ \frac{A}{DE} + \frac{B}{FG} + \frac{C}{HJ} = 1 \]

   where each letter represents a different digit and none of them is zero. DE, FG, and HJ are each two-digit integers, e.g., DE = 16 * D + E. We want the solution in which A < B < C.

   —Technology Review

**DOUBLE BONUS.** Consider an infinite square lattice of points with integer coordinates on a plane. If two lattice points are chosen at random, what is the probability that the straight-line segment joining the two lattice points will pass through any other lattice points?

—The Joy of Pi by David Blatner

Mail your answers to any or all of the Brain Ticklers to: Jim Froula, Tau Beta Pi, P. O. Box 2697, Knoxville, TN 37901-2697 or email plain text to: BrainTicklers@tbp.org. The cutoff date for entries to the Fall column is around early January. The method of solution is not necessary. We welcome any interesting new problems that might be suitable for use in the column. The Double Bonus is not graded. Jim will forward your entries to the judges, who are:

H. G. McIlvried III, PA F ’53;
D. A. Dechman, TX A ’57;
F. J. Tydeman, CA A ’73;
and the columnist for this issue, J. L. Bradshaw, PA A ’82.