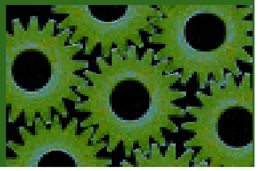

Brain Ticklers

RESULTS FROM WINTER 2005

Perfect

*	Fable, Scott E.	CA	Τ	'96
*	Prince, Lawrence R.	CT	Β	'91
	Schmidt, V. Hugo	WA	Β	'51
	Spong, Robert N.	UT	Α	'58
	Strong, Michael D.	PA	Α	'84
*	Thaller, David B.	MA	Β	'93
*	Weinstein, Stephen A.	NY	Γ	'96
	Yee, David G.	NJ	Β	'04

Other

	Aron, Gert	IA	Β	'58
	Bachmann, David E.	MO	Β	'72
*	Baines, Elliot "Chip" A., Jr.	NY	Δ	'78
	Baldwin, Scott K.	MA	Ε	'91
	Braña, Francisco J., Jr.	PR	Α	'74
*	Brule, John D.	MI	Β	'49
*	Christenson, Ryan C.	UT	Β	'93
*	Couillard, J. Gregory	IL	Α	'89
	Creutz, Michael J.	CA	Β	'66
	Creutz, Edward C.	PA	Γ	'36
*	Eddy, Sarah J.	DC	Β	'01
*	Garnett, James M., III	MS	Α	'65
	Griggs, James L., Jr.	OH	Α	'56
	Johnson, Roger W.	MN	Α	'79
	Jones, Donlan F.	CA	Ζ	'52
	Larson, M. Rhett	KS	Γ	'04
*	Nabutovsky, Joseph	Father of member		
	Post, Irving G.	PA	Ε	'58
*	Powell, Scott F.	NC	Γ	'97
	Purdy, Herbert W., III	PA	Ε	'59
*	Quintana, Juan S.	OH	Θ	'62
*	Rentz, Peter E.	IN	Α	'55
	Snyder, M. Duane	IA	Β	'63
*	Stribling, Jeffrey R.	CA	Α	'92
	Valko, Andrew G.	PA	Λ	'80
	VanShaar, Steven R.	UT	Γ	'00
*	Voellinger, Edward J.	Non-member		
	Volder, Jack E.	TX	Β	'49
	Wisniowski, Tomasz	WI	Ε	'06

* *Denotes correct bonus solution*

FALL ENTRY for "Other"

Peralta-Maninat, Alfredo J.	MA	Β	'54

WINTER REVIEW

Problem 5 about the 3-D tic-tac-toe was the hardest problem (even harder than the Bonus), with less than half of the answers being correct.

Correction: The solution for the Bonus question has errors that were introduced when laying out the magazine.

1DH should not have a "T"
3DH should have a "K"
5DH should not have an "H"

SPRING SOLUTIONS

Readers' entries for the Spring problems will be acknowledged in the Fall BENT. Meanwhile, here are the answers:

1 The number of sides of the flower bed is 42. Consider a triangular segment of the flower bed and the bordering trapezoidal stone. Let r_i and r_o be the radii of the inscribed and circumscribed circles of the stone. Then, $r_i = kr_o$, where $k = 19/22$. The altitude of this triangle equals $r_i + h = r_o\cos\theta$, where h is the altitude of a stone, $\theta = \pi/N$ is half the apex angle, and N is the number of sides of the flower bed. Now, the lengths of the two bases of the trapezoidal stone are given by $2r_i\tan\theta = 2kr_o\tan\theta$ and $2r_o\sin\theta$. Therefore, the area of the walk is given by $A_w = Nr_o^2(\cos\theta - k)(k\tan\theta + \sin\theta)$, and the area of the flower bed is given by $A_f = Nk^2r_o^2\tan\theta$. Therefore, since, $A_f/A_w = 3$, we have, $k^2\tan\theta/[(\tan\theta - k)(k\tan\theta + \sin\theta)] = 3$. Trying various values for N, shows that $N = 42$ gives a value of 3.0009 for the ratio of the areas, which is closer than either $N = 41$ or $N = 43$.

2 To make it a fair game, the player should pay the house \$1.76. It is clear from the symmetry of the game that the expected value of the dollars added to and taken from the pot by the player is zero. Therefore, we need only consider the expected value of winning the pot. For the player to win the pot, the number of HH tosses must equal the number of TT tosses. The probability of this happening is given by $p(n) = 10!(0.25^{2n})(0.5^{10-2n})/[(n!)^2(10-2n)!]$. Therefore, the expected value of the game is $E = 10\Sigma p(n)$, where n goes from 0 to 5. This gives $E = 1.762$. Therefore, the player should pay \$1.76.

3 If the mother chased Bill first, it would still take five hours. Let t_A be the time it takes to catch Ann and t_B be the time it takes to catch Bill once mom passes home. Then, $t_A = (1 + t_A)r_A/10$ and $t_B = (1 + 2t_A + t_B)r_B/10$, where r_A and r_B are Ann's and Bill's speeds. Solving these equations gives $t_A = r_A/(10 - r_A)$ and $t_B = (10r_B + r_Ar_B)/[(10 - r_A)(10 - r_B)]$. Now, $2(t_A + t_B) = 5$. Adding t_A and t_B and simplifying gives $10(r_A + r_B)/[(10 - r_A)(10 - r_B)] = 2.5$. Using the same reasoning for the second case where mom chases Bill first, we find $t_B = r_B/(10 - r_B)$ and $t_A = (10r_A + r_Ar_B)/[(10 - r_A)(10 - r_B)]$. Adding, we get $t_A + t_B = 10(r_A + r_B)/[(10 - r_A)(10 - r_B)]$, but this is equal to 2.5. Therefore, the length of the trip is the same regardless of whether mom chases Ann first or Bill first.

4 M = 6,174. Let D = 1,000P + 100Q + 10R + S. Then, A = 1,000S + 100R + 10Q + P, and M = D – A = 999P + 90Q – 90R – 999S = 999(P – S) + 90(Q – R). Because all the digits are different, 2 < P – S < 9, and 0 < Q – R ≤ P – S – 2. Therefore, possible values for 999(P – S) are 2,997; 3,996; 4,995; 5,994; 6,993; and 7,992. For each of these, we calculate 999(P – S) + 90(Q – R) for all possible values of Q – R and check whether the largest digit minus the smallest digit equals P – S and whether the difference of the other two digits equals Q – R. Only for P – S = 6 and Q – R = 2 do we get a match.

5 The prime $P = 2^{24,036,583} - 1$ has 7,235,733 digits, of which the first three are 299 and the last three are 407. The first part of the problem is easy. It merely requires using a sufficiently accurate value for log2 = 0.301029995. Now, 24,036,583log2 = 7,235,732.476. Since $\log^{-1}2.476 = 299.4$, P has 7,235,733 digits, starting with 299. To find the last three digits, we make use of a theorem due to Euler. He showed that for any two relatively prime numbers, $a^{\phi(b)} \equiv 1 \ (mod\ b)$, where $\phi(b)$ is Euler's phi function. For a power of a prime, p^n, $\phi(p^n) = p^{n-1}(p - 1)$. Let $a = 2$ and $b = 5^3 = 125$. Then, $\phi(125) \equiv 5^2(5 - 1) = 100$, and $2^{100} \equiv 1 \pmod{125}$, or raising both sides to the nth power, $2^{100n} \equiv 1 \pmod{125}$. Therefore, $2^{100n} \equiv 1 + 125k$. Multiplying by 8 gives $8(2^{100n}) = 2^{100n+3} \equiv 8 + 1{,}000k$. Therefore, $2^{100n+3} \equiv 8 \pmod{1{,}000}$. Since $24{,}036{,}583 = 240{,}365(100) + 80 + 3$, $2^{24,036,583} \equiv 8(2^{80}) \pmod{1{,}000}$. Now, $2^{80} = (2^{10})^8 \equiv 24^8 = 576^4 \equiv 176 \pmod{1{,}000}$. Therefore, $8(2^{80}) \equiv 8(176) \equiv 408 \pmod{1{,}000}$, so the last three digits are 408 – 1 = 407.

Bonus The ellipse has a 63.13% probability of falling on one of the parallel lines. Construct an ellipse with major semiaxis a along the x-axis, minor semiaxis b along the y-axis, and center at the origin. The equation of this ellipse is $x^2/a^2 + y^2/b^2 = 1$. Now, in the fourth quadrant, construct a line through the origin at an angle θ to the x-axis and intersecting the ellipse at point A. The slope of this line is -tanθ. In the first quadrant, construct a tangent to the ellipse parallel to line OA. The slope of the ellipse at any point is given by $dy/dx = -b^2x/a^2y$. Thus, $b^2x/a^2y = \tan\theta$. Let the point of tangency be (x_t, y_t). Solving the two equations, $b^2x_t/a^2y_t = \tan\theta$ and $x_t^2/a^2 + y_t^2/b^2 = 1$, simultaneously we get $x_t = a^2\tan\theta/(a^2\tan^2\theta + b^2)^{0.5}$ and $y_t = b^2/(a^2\tan^2\theta + b^2)^{0.5}$. Now, the equation of the tangent is $y = c - x\tan\theta$. Substituting the values for x_t and y_t and solving for c, we get $c = (a^2\tan^2\theta + b^2)^{0.5}$. If the ellipse is rotated counterclockwise by an angle θ, then the tangent line will be parallel to the lines in the grid. Let s be the perpendicular distance between the tangent and OA. If the center of the ellipse is closer than s to one of the grid lines, then the ellipse will touch the line. Therefore, if the ellipse falls with its major axis at an angle θ to the grid lines, then the probability that it will touch the line is $2s/D$, where D is the distance between grid lines. (The 2 arises because the center of the ellipse will be between two grid lines, and the ellipse could touch either line.) The probability that the ellipse will fall at an angle θ is $d\theta/(\pi/2) = 2d\theta/\pi$. Therefore, the probability P that the ellipse touches a grid line equals the integral from 0 to $\pi/2$ of $4s\,d\theta$, but $s = c\cos\theta = \cos\theta(a^2\tan^2\theta + b^2)^{0.5} = (a^2\sin^2\theta + b^2\cos^2\theta)^{0.5}$. Substituting $\sin^2\theta = 1 - \cos^2\theta$ and simplifying gives $s = a(1 - k^2\cos\theta)^{0.5}$, where $k = (a^2 - b^2)^{0.5}/a$. Thus, P equals $4a/\pi D$ times the integral from 0 to $\pi/2$ of $(1 - k^2\cos\pi)^{0.5}$. Since sin and cos take the same values over the range 0 to $\pi/2$, we can change cos to sin in the integral. When we do this, we see that we have the complete elliptic integral of the second kind, represented by $E(k, \pi/2)$, whose values are tabulated. Thus, $P = 4aE(k, \pi/2)/\pi D$. For the given conditions ($a = 0.375$, $b = 0.25$, $D = 1$), $E(k, \pi/2) = 1.3221$, and $P = 4(0.375)(1.3221)/3.14159 = 0.63127$. In general, the desired probability is the perimeter of the ellipse divided by the perimeter of a circle just tangent to two adjacent grid lines.

Computer Bonus
$10^{33} = 2^{33} \times 5^{33}$ = 8,589,934,592 x 116,415,321,826,934,814,453,125 is the highest power of 10 that can be expressed as the product of two factors without using any zeros, of which we are aware.

SUMMER PROBLEMS

1 Five chess fanatics always manage to fit in a round-robin before their train reaches Waterloo. Yesterday there were no ties in the final order (2 points for win, 1 for draw, 0 for loss). Alapin won his game against the only person who took a game off Bird. Catalan was the only person to lose a game against the financier, who was the only person to lose a game to Dunst. Alapin finished below the ghost-writer. The interpreter scored only 1 point total. The journalist fared worse than the hairdresser. As for Evans, you can work that out for yourself. What is each person's score and occupation?

—*New Scientist*

2 Consider a set of N distinct positive integers of your choice, the sum of any K of which is prime. What is the maximum possible value of N for K = 2, 3, 4, and 5?

—*Technology Review*

3 Frances was trying to remember how to arrange the numbers 1 to 9 in a magic square. She tried:

7	8	9
6	1	2
5	4	3

This was her first shot, and, as you can see, rather than getting the three columns, three rows, and two diagonals to all add to the same total, they all summed to different totals. "Look," she said, "I have invented an anti-magic square." What is the smallest anti-magic square, consisting of nine positive, not necessarily all different, integers? In judging smallest, the first criterion is to minimize the largest integer used, the second criterion is to minimize the total of all nine integers, and the third criterion is to minimize the sum of the first row.

— Anti-Magic Square
by Stephen Ainley

4 If N is a positive integer, what rational values may be assumed by $R = \sqrt{(N/7)^2 + 5{,}036}$?

—*American Mathematical Monthly*

5 The Parliament of Puevigi plans to divide the population into 10 income groups and to average the wealth between each pair of neighboring groups. The averaging will be done as nine distinct steps done in order: starting with the two lowest (poorest) groups, then groups two and three, ending with the top two groups. An amendment to this plan proposes to work from the top down. Which plan, if either, should the richest groups prefer? The poorest groups prefer?

—Litton Industries
"Problematical Recreations"

Bonus At a summer camp, three campers show up at the mess hall for their share of the lemonade. One camper has a 10 Liter container, the second has a 6 L container, and the third has a pail of unknown capacity, but greater than 2 L. The mess-hall attendant has a 15 L container full of lemonade, and she needs to distribute exactly 8 L to the 10 L container, exactly 5 L to the 6 L container, and exactly 2 L to the pail. If no other containers are available, how does she accomplish this? Present your answer as a table with four columns labeled 15L, 10L, 6L, and Pail, with each row giving the contents of the containers after each transfer. For example, the first row is (15, 0, 0, 0); now if the 10 L container is filled, the second row would be (5, 10, 0, 0), etc. Containers cannot be tipped at an angle to provide half or any other fraction of their volume. We want the shortest sequence of exchanges that will work.

—Dr. Richard I. Hess, CA *B* '62

Double Bonus Given three circles, each with a different radius, externally tangent to each other. For each pair of circles, construct the two lines that are tangent to both circles, but not the tangent line where the two circles meet. Each of these pairs of lines will meet at a point. Prove that the three meeting points lie on a straight line.

— Daryl Cooper

Postal mail your answers to any or all of the Brain Ticklers to: Jim Froula, Tau Beta Pi, P. O. Box 2697, Knoxville, TN 37901-2697 or email plain text to: *BrainTicklers@tbp.org*. The cutoff date for entries to the Summer column is the appearance of the Fall BENT in early October. The method of solution is not necessary. We also welcome any interesting new problems that may be suitable for use in the column. The Double Bonus is not graded. Jim will forward your entries to the judges, who are: H.G. McIlvried III, PA *G* '53; D.A. Dechman, TX *A* '57; J.L. Bradshaw, PA *A* '82; and the columnist for this issue,

—F.J. Tydeman, CA *D* '73