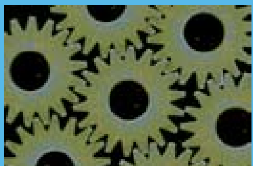

Brain Ticklers

RESULTS FROM FALL 2009

Perfect

Couillard, J. Gregory	IL	A '89
Fenstermacher, T. Edward	MD	B '80
Kimsey, David B.	AL	A '71
Lewandowski, Jerold J.	NY	Γ '92

Other

Alexander, Jay A.	IL	Γ '86
Aron, Gert	IA	B '58
Bachmann, David E.	MO	B '72
Bernacki, Stephen E.	MA	A '70
Bertrand, Richard M.	WI	B '73
Brook, John W.	NY	Z '56
Brule, John D.	MI	B '49
Davis, John H.	OH	Γ '60
Doniger, Kenneth J.	CA	A '77
Handley, Vernon K.	GA	A '86
Jones, Donlan F.	CA	Z '52
Quintana, Juan S.	OH	Θ '62
Rasbold, J. Charles "Chuck"	OH	A '83
Sigillito, Vincent G.	MD	B '58
Spong, Robert N.	UT	A '58
Spring, Gary S.	MA	Z '82
*Stribling, Jeffrey R.	CA	A '92
Strong, Michael D.	PA	A '84
Tessier, Thomas M.	MA	A '90
Thaller, David B.	MA	B '93
Voellinger, Edward J.	Non-member	
Weinstein, Stephen A.	NY	Γ '96
Wolff, Nicholas L.	NE	A '00

* Denotes correct bonus solution

FALL REVIEW

The most difficult regular problems were No. 1, about scoring in a cross-country match, and No. 4, about the distribution of random points in a hemisphere. The bonus problem, about the frequency of oscillation of a circular pendulum, appears to be the most difficult problem we have had for some time, with only one correct answer. The judges also had a great deal of difficulty with that problem.

WINTER SOLUTIONS

The Winter entries will be acknowledged in the Summer Bent, but, meanwhile, here are the answers. Notice that all the Winter ticklers, including the computer bonus, can be solved without a computer.

1 The maximum number of drops to test the bowling balls is 14. A building of three floors requires two drops. Try the 2nd floor first; if the ball breaks, try the 1st floor next; if it doesn't, try the 3rd floor. A building of six floors requires three drops. Try the 3rd floor first; if the ball breaks, try the 1st floor and then the 2nd floor; if the ball doesn't break, try the top three floors as described above. A building of ten floors requires four drops, starting with a drop from the 4th floor. If the ball breaks, try floors 1, 2, and 3 in order. If the ball doesn't break, try the top six floors using the above strategy. In general, N drops are required for $N(N + 1)/2$ floors (the sum of 1 through N); start on the N^{th} floor; if the ball breaks, try floors 1 through N-1 in order; if the ball doesn't break, proceed as indicated above with the top $N(N - 1)/2$ floors. For $N = 14$, the maximum number of floors is 105. For 100 floors, drop the first ball successively from floors 14, 27, 39, 50, 60, 69, 77, 84, 90, 95, 99, 100 until it breaks, and then proceed as described above.

2 It will take 462 days to exhaust all possibilities, as shown by the following table, in which C(n, m) is the combinations of n things taken m at a time and P(n, m) is the permutations of n things taken m at a time.

Distribution of Kinds of Donuts	Equation	No. of Ways
1,1,1,1,1,7	C(6,1)	6
1,1,1,1,2,6	P(6,2)	30
1,1,1,1,3,5	P(6,2)	30
1,1,1,1,4,4	C(6,2)	15
1,1,1,2,2,5	6C(5,2)	60
1,1,1,2,3,4	P(6,3)	120
1,1,1,3,3,3	C(6,3)	20
1,1,2,2,2,4	6C(5,3)	60
1,1,2,2,3,3	C(6,2)C(4,2)	90
1,2,2,2,2,3	P(6,2)	30
2,2,2,2,2,2	C(6,6)	1
Total		462

3 The smallest and largest values of N such that exactly half the integers less than or equal to N contain a digit 5 are 590,488 and 5,314,408. The integers 0-9 contain 1 integer with a 5; 0-99 contains 19 such integers; 0-999 contains 271; 0-9,999 contains 3,439; 0-99,999 contains 40,951; 0-999,999 contains 468,559; and 0-9,999,999 contains 5,217,031. These numbers can be used to bracket values of N. The first N for which exactly half the integers contain a 5 occurs between 500,000 and 600,000. So $499{,}999 + x = 2[5(40{,}951) + x]$, or $x = 90{,}489$, and $N = 590{,}488$. The next value occurs at $2[5(40{,}951) + 100{,}000 + 3(3439) + 2] = 2(315{,}112) = 630{,}224$, and the next, two integers later at 630,226. The next occurs between 650,000 and 660,000, so $649{,}999 + x = 2[5(40{,}951) + 100{,}000 + 5(3439) + x]$, or $x = 6{,}099$, so $N = 656{,}098$. The next N is at $2[5(40{,}951) + 100{,}000 + 6(3439) + 271 + 6(19) + 100 + 7 + 10] = 2(335{,}891) = 671{,}782$. Finally, the largest value of N occurs between 5,000,000 and 6,000,000, so $4{,}999{,}999 + x = 2[5(468{,}559) + x]$, or $x = 314{,}409$, so $N = 5{,}314{,}408$. From this point on, there are more integers containing a 5 than without.

4 SENT = 3019. The only possible values for TEN, which is 1 more than a perfect square divisible by 9, are 145, 325, and 901 (730 is eliminated because N cannot be 0). Since NINETY is divisible by 9, the sum of its digits must also be divisible by 9. Therefore, if TEN = 145, then I + Y = 3 or 12; if TEN = 325, then I + Y = 12; and if TEN = 901, then I + Y = 7. Trying all possibilities and realizing that SIX is close to the difference between the square roots of NINETY and TEN results in SIX = 358, TEN = 901, and NINETY = 151,092 being the only solution.

5 It helps to start by listing all 22 three-digit squares. Next, construct all the trial *squares* you can with each side being one of the three-digit squares (there are 7). Two of these must fit into the final grid. Some are eliminated as they would require a square ending in 2 or 8; and some pairs can't be used together as this would require using the same square more than once. After a bit of trial and error, the following unique solution results.

8	■	1	6	9	■	2	2	5	■	1
4	8	4	■	6	2	5	■	7	2	9
1	■	4	4	1	■	6	7	6	■	6

Bonus. The resistance across a body diagonal of a six-dimension hypercube, whose edges are 1-ohm resisters, is 13/30 ohm. You can get an idea of how to proceed by drawing schematic diagrams of the two-, three-, and four-dimensional cases. You will see that, if you connect points of equal potential, the hypercube can be considered to consist of D layers with $(D - i)C(D, i)$ resistors in the ith layer, where $C(n, m)$ is the combinations of n things taken m at a time, and i varies from 0 to $D - 1$. The situation for the six-dimensional case is shown in the following table.

Layer	Nodes	Resistors			Resistance, ohm
		In	Out	Number	
1	1	0	6	6	1/6
2	6	1	5	30	1/30
3	15	2	4	60	1/60
4	20	3	3	60	1/60
5	15	4	2	30	1/30
6	6	5	1	6	1/6
	1	6	0		
Total	64			192	13/30

In general, the resistance across a body diagonal of a D-dimensional hypercube is given by the sum from $i = 0$ to $D - 1$ of $1/[(D-i)C(D, i)]$. The results for $D = 2, 3, 4, 5$, and 6 are 1, 5/6, 2/3, 8/15, and 13/30 ohm, respectively.

Computer Bonus. The next four integers, larger than 36, that are both triangular numbers and perfect squares are 1,225; 41,616; 1,413,721; and 48,024,900. Although this problem can be solved by computer, there is also an analytical solution. Let the number be N. Then $N = n(n+1)/2 = m^2$, or $n^2 + n = 2m^2$. Multiplying by 4 and adding 1 to each side gives $(2n+1)^2 = 2(2m)^2 + 1$, or $p^2 - 2q^2 = 1$, where $p = n+1$ and $q = 2m$. This is the familiar Pell equation, whose method of solution is well known (see almost any book on number theory). Given the first two solutions ($N = 1, n = 1, m = 1, p = 2(1)+1 = 3, q = 2(1) = 2$, and $N = 36, n = 8, m = 6, p = 17, q = 12$), other solutions can be generated using the recurrence relationship, $q_{n+1} = 6q_n - q_{n-1}$. Thus, we have $q_3 = 6(12) - 2 = 70$, $N = (70/2)^2 = 1{,}225$; $q_4 = 6(70) - 12 = 408$, $N = (408/2)^2 = 41{,}616$; $q_5 = 6(408) - 70 = 2{,}378$, $N = (2{,}378/2)^2 = 1{,}413{,}721$; and $q_6 = 6(2{,}378) - 408 = 13{,}860$, $N = (13{,}860/2)^2 = 48{,}024{,}900$.

NEW SPRING PROBLEMS

1 Find a 3x3 magic square, consisting of nine different, although not necessarily consecutive, integers less than 100, that has the following property. When the numbers in the magic square are replaced by the number of letters in their English names, the square is still magic with nine different integers. For example, 10 would be replaced by 3 (ten has 3 letters) and 23 would be replaced by 11 (twenty-three has 11 letters). A magic square is a square array of all different integers in which each row, column, and major diagonal sum to the same constant.

—*50 Mathematical Ideas You Really Need to Know* by Tony Crilly

2 Ann and Beth live on Elm Street, which has houses numbered 1 through 99, but neither knows the other's house number. Zack lives in another house on Elm Street, and the women are eager to discover his number. Ann asks Zack two questions: (1) is your number a perfect square, and (2) is it greater than 50? Having heard Zack's answers, she declares she knows his number, but she is wrong because only Zack's second answer is true. Beth, not having heard any of this, asks Zack two different questions: (1) is your number a perfect cube, and (2) is it greater than 25? Like Ann, after hearing Zack's answers, she declares she knows his number, but she is wrong because again only Zack's second answer is true. Given the further information that Zack's number is less than both Ann's and Beth's numbers and that the sum of their three numbers is twice a perfect square, what are Ann's, Beth's, and Zack's house numbers?

—*Brain Puzzlers Delight* by E.R. Emmet

3 In a certain gambling institution, the game of Lucky-10 is played. The house provides a pot of $10. The player then tosses two coins simultaneously. If two heads appear, the player takes $1 from the pot; if two tails appear, the player adds $1 to the pot; and if a head and a tail appear, the player neither adds nor takes from the pot. After ten tosses (including draws), if there is exactly $10 in the pot, the player takes it. Otherwise, the house gets whatever is in the pot. How much should the player initially pay the house (to the nearest penny) to make this a fair game, that is, with expected winnings of 0?

— Joseph-Louis Lagrange (1736-1813)

4 How many digits long is 100,000!?

—*Prime Numbers, the Most Mysterious Figures in Math* by David Wells

5 Solve the following cryptarithm, in which none of the letters is zero.

AB/CD + EF/GHJ = A

Each of the letters represents a different digit, and each digit is a different letter.

—Joseph Nabutovsky

Bonus. A hula-hoop is supported on a knife edge so that it hangs in a vertical plane. It is given a small push and set into back and forth oscillatory motion in the vertical plane. Later, the bottom half of the hula-hoop is cut off so that exactly one quarter of the hoop is on each side of the knife edge, and the half hoop is set into oscillatory motion in the vertical plane. What is the ratio of the period of oscillation of the half hoop to that of the whole hoop?

—*Mad About Physics* by Christopher P. Jargodzki and Franklin Potter

Computer Bonus. Sequences of four consecutive primes whose last digits are 1, 3, 7, and 9, but whose other digits are the same, are not common. The first two such sequence are (11, 13, 17, 19) and (101, 103, 107, 109). What is the thousandth such sequence?

–Don A. Dechman, *TX A '57*

Send your answers to any or all of the Spring Brain Ticklers to Jim Froula, Tau Beta Pi, P.O. Box 2697, Knoxville, TN 37901-2697, or email to <u>BrainTicklers@tbp.org</u>, plain text only. The cutoff date for entries to the Spring column is the appearance of the Summer BENT in late June. It is not necessary to include the method of solution. The computer bonus is not graded. The judges welcome any interesting problems that might be suitable for the column. Jim will forward your entries to the judges who are **F.J. Tydeman,** *CA Δ '73*; **D.A. Dechman,** *TX A '57*; **J.L. Bradshaw,** *PA A '82*; and the columnist for this issue, **H.G. McIlvried, III,** *PA Γ '53*.